On the Derivation of Bell-Plesset Instability

Stephen Charles
Supervisor: Dr. Matthew Scase

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Abstract

The works of Rayleigh, Taylor, Plesset, Bell, Binnie, Birkhoff, Fisher, Prosperitti, Lin and Amendt form the basis of this review, concerning *Bell-Plesset Instability.*[1–13] Amendt's treatment has been expanded to include viscous effects via Joseph's method, where a form of Bernoulli's equation is used, incorporating compressibility, derived from the Navier Stokes formulation.

I Introduction

The basis of investigation for this paper stems from the seminal paper of Plesset,[5] who was the first to formally publish material of bubble collapse, specifically concerning the perturbation of a collapsing spherical interface. The description of this process has since been coined *Bell-Plesset* instability. He drew inspiration from a textbook by the name of *Underwater Explosions*,[14] which dealt solely with the topic of the title. His determination from this analysis found that bubbles do not always retain their spherical symmetry.

Prosperitti[15] worked closely with Plesset, and expanded on his work by introducing a viscous correction to the liquid. His approach involved taking the curl of the dynamical equations to express the viscosity through vorticity. My own approach is in conflict to this derivation, and through comparison with experimentation I hope to explore validity of both approaches. Prosperitti assumes the viscosity arises through vorticity, and I express the viscous correction by assuming zero vorticity. Of note here is the similarity by which the two methods have with respect to their proportionality. They differ only slightly by their respective modal Atwood numbers, namely the terms in the case of a bubble where $\rho_1 = 0$

$$\begin{aligned} \mathbb{V}_1 &= -2\nu_2(n-1)(n+2); \\ \mathbb{V}_2 &= -4\nu_2(n^2+4n+1); \end{aligned} \qquad \ldots(1)$$

in contrast to Prosperitti's approach

$$\begin{aligned} \mathbb{V}_1 &= 2\nu_2(2n+1)(n+2); \\ \mathbb{V}_2 &= 2\nu_2(n-1)(n+2), \end{aligned} \qquad \ldots(2)$$

of which I suspect shows more consistent promise. Physically, I'm not too sure whether a fluid with negative viscosity is valid in reality via method (1)!

In practical application this study is important in the field of sonochemistry. The extreme rate of collapse confines intense energy focused into a singularity, which is responsible for high temperatures and subsequent reactions. If the bubble collapse is not spherical, the consensus is that this focusing of energy is less intense, and the maximum achievable temperatures are correspondingly lower; therefore determination of interfacial stability would allow predictions on the influence this boundary has on chemical production.

The main pursuit then, is to first derive from scratch Plesset's approach, and then follow with the viscous corrections of Prosperitti and my own. The investigation will conclude with the comparison of the two approaches, and whether my simplified approach through assumption of zero vorticity is valid. The following page lists both of the final results of both methods. So far I have completed the derivation of Plesset[5], Amendt[13], Fisher[10] and Bell[8], of which only Plesset's is shown here. All were verified by both hand and the Mathematica code shown in section (D). Amendt[13] followed all of these approaches with a modified potential which is regular at the origin, but analysis shows that in the limits of a bubble perturbation, the original work of Plesset suffices in our investigation. This is demonstrated later in section (IV), and justifies the use of Plesset's potential. Derivation of Prosperitti's approach is a work in progress.

The following pages first derive Plesset's seminal paper on the topic, with a comparison of his approach versus Amendt's, concluding in the justification for the potential used in our specific case. Next a substitution is made in the final result to solve the result analytically via a *WKBJ* approximation, giving us a view of the proportionality relationship of the motion over time. This approximation should allow us, to a reasonable error, to verify assumptions made in Prosperitti's approach on the integral parts of his solution; he assumes that these have a small contribution compared to the differential parts, which is a claim worth verifying.

My Approach

Incompressible Bell-Plesset Instability with Viscous Terms and Surface Tension

$$\ddot{a} + \dot{a}\left(\frac{3\dot{R}}{R} + \frac{\mathbb{V}_1}{R^2}\right) + a\left(\frac{\dot{R}\mathbb{V}_2}{R^3} + \frac{\ddot{R}\mathbb{A}_1}{R} + \frac{\mathbb{S}_1}{R^3}\right) = 0. \qquad \ldots(3)$$

Modal Atwood Numbers

$$\begin{aligned} \mathbb{S}_1 &= \frac{(n-1)(n+1)(n+2)\sigma}{\rho_2}; & \mathbb{V}_1 &= -2\nu_2(n-1)(n+2); \\ \mathbb{A}_1 &= -(n-1); & \mathbb{V}_2 &= -4\nu_2(n^2+4n+1); \end{aligned} \qquad \ldots(4)$$

Prosperitti's Approach

Incompressible Bell-Plesset Instability with Viscous Terms and Surface Tension

$$\ddot{a} + \dot{a}\left(\frac{3\dot{R}}{R} + \frac{\mathbb{V}_1}{R^2}\right) + a\left(\frac{\dot{R}\mathbb{V}_2}{R^3} + \frac{\ddot{R}\mathbb{A}_1}{R} + \frac{\mathbb{S}_1}{R^3}\right) = 0. \qquad \ldots(5)$$

Modal Atwood Numbers

$$\begin{aligned} \mathbb{S}_1 &= \frac{(n-1)(n+1)(n+2)\sigma}{\rho_2}; & \mathbb{V}_1 &= 2\nu_2(2n+1)(n+2); \\ \mathbb{A}_1 &= -(n-1); & \mathbb{V}_2 &= 2\nu_2(n-1)(n+2), \end{aligned} \qquad \ldots(6)$$

List of Symbols

a	Distortion Amplitude		
R	Interface Radius	ϕ	Velocity Potential
F	Time Dependent Density	μ	Dynamic Viscosity
Y_n	Spherical Harmonic	ν	Kinematic Viscosity
$\mathbb{A}_n$	Modal Atwood Number	ρ	Density
$\mathbb{V}_n$	Viscous Terms Atwood Number	σ	Surface Tension
$\mathbb{S}_n$	Surface Tension Term Atwood Number		

II Derivation of Plessets Approach

Solution to the Incompressible Stability Problem

The analysis presented here follows the approach of Plesset.[5] A fluid of density ρ_1 is contained within a sphere of radius R; a fluid of density ρ_2 occupies the region exterior to this sphere. The distortion, or ripple at the interface is denoted by r_s. Radial position within the liquid will be denoted by the distance r, from the center of the bubble. σ is the surface tension and $\mathbb{A}_n$ is the modal Atwood number (density ratio) of the instability. The ripple is contained by the regions $r > R$, inside the ripple, and $r < R$, outside of the ripple.

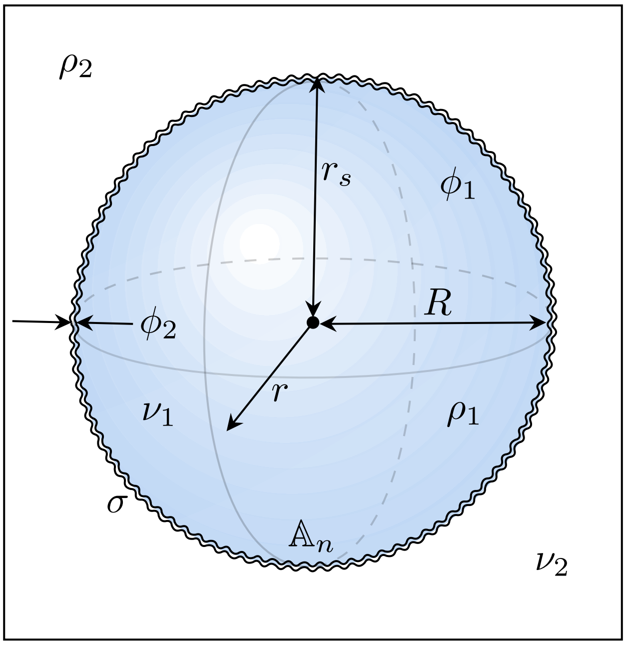

Figure 1: *Notation for Bell-Plesset Instability*

Velocity Potential ϕ

As the flow is irrotational, there exists a velocity potential ϕ, which satisfies Laplace's equation $\nabla^2\phi = 0$ in spherical coordinates. If ϕ is independent of θ, φ we have the velocity potential

$$\phi = \frac{A}{r} + B. \quad \ldots(7)$$

If ϕ is independent of φ:

$$\phi = \left(Ar^n + \frac{B}{r^{n+1}}\right) Y_n^m, \quad \ldots(8)$$

where Y_n^m is a spherical harmonic. We take $m = 0$ as its likely that the sphere will have an axis of symmetry and the radial dependence is also independent of m. The origin of a spherical coordinate system is taken at the center of the spherical interface $R(t)$. When the interface is strictly spherical, the velocity potential is

$$\phi = \frac{R^2\dot{R}}{r}, \quad \ldots(9)$$

where the radial velocity at the point r in the fluid is $\frac{\partial\phi}{\partial r}$. This potential implies a source or sink at the origin depending on the sign of $\dot{R}$. The stability of the spherical interface will be established by considering whether a distortion of the interface of small amplitude grows or diminishes. We consider a distortion at the interface from R to r_s, where

$$r_s = R + aY_n. \quad \ldots(10)$$

Y_n is a spherical harmonic of degree n and a is a function of time t such that we have the relationship

$$|a(t)| \ll R(t),$$

where we clarify that

$$\frac{|a(t)|}{R(t)} \ll 1 = \varepsilon.$$

The stability analysis will be limited to $\mathcal{O}(\varepsilon)$. To this order, the fluid particle velocity at the interface in the radial direction is given by

$$u = \frac{\partial r_s}{\partial t} = -\frac{\partial \phi}{\partial r}\bigg|_{r=r_s} = \dot{R} + \dot{a}Y_n. \qquad \ldots(11)$$

Across the interface the normal component of the fluid velocity must be continuous. The difference between the normal component of the fluid velocity at the interface and the radial velocity u is of second order in a so that the boundary condition is satisfied by the requirement of continuity of u across the interface. If one chooses a potential which corresponds to a disturbance which decreases away from the interface in both the inward and outward directions, the potential then becomes

$$\phi = \begin{cases} \phi_1 = \dfrac{R^2\dot{R}}{r} + b_1 r^n Y_n, & \text{for } r < R; \\ \phi_2 = \dfrac{R^2\dot{R}}{r} + b_2 \dfrac{Y_n}{r^{n+1}}, & \text{for } r > R. \end{cases} \qquad \ldots(12)$$

With the condition that continuity is satisfied such that b_1 and b_2 are found via

$$-\frac{\partial \phi_1}{\partial r}\bigg|_{r=r_s} = -\frac{\partial \phi_2}{\partial r}\bigg|_{r=r_s} = \dot{R} + \dot{a}Y_n. \qquad \ldots(13)$$

To find b_1 and b_2

Solving first the velocity potential

$$-\frac{\partial \phi_1}{\partial r} = \frac{R^2\dot{R}}{r^2} - nb_1 r^{n-1} Y_n = \dot{R} + \dot{a}Y_n,$$

and evaluating at $r = r_s$ from Eq. 13, we have that

$$-\frac{\partial \phi_1}{\partial r}\bigg|_{r=r_s} = \frac{R^2\dot{R}}{(R + aY_n)^2} - nb_1(R + aY_n)^{n-1}Y_n = \dot{R} + \dot{a}Y_n.$$

Rearranging for b_1, we find at $\mathcal{O}(\varepsilon)$,

$$\begin{aligned} b_1 &= -\frac{1}{nY_n(R + aY_n)^{n+1}}\left(\left(\dot{R} + \dot{a}Y_n\right)\left(R^2 + 2RaY_n + a^2Y_n^2\right) - R^2\dot{R}\right) \\ &= -\frac{1}{nY_nR^{n+1}}\left(2R\dot{R}aY_n + \dot{a}Y_nR^2 +\right) + \mathcal{O}\left(\varepsilon^2\right) \\ &= -\frac{1}{nR^{n-1}}\left(\dot{a} + 2a\frac{\dot{R}}{R}\right) + \mathcal{O}\left(\varepsilon^2\right). \end{aligned} \qquad \ldots(14)$$

Similarly for the potential

$$-\frac{\partial \phi_2}{\partial r} = \frac{R^2\dot{R}}{r^2} + (n+1)b_2 r^{-(n+2)} Y_n = \dot{R} + \dot{a}Y_n,$$

and evaluating at $r = r_s$ from Eq. 13, we have that

$$-\frac{\partial \phi_2}{\partial r}\bigg|_{r=r_s} = \frac{R^2\dot{R}}{(R + aY_n)^2} + (n+1)b_2(R + aY_n)^{-(n+2)}Y_n = \dot{R} + \dot{a}Y_n.$$

Rearranging for b_2, we find at $\mathcal{O}(\varepsilon)$,

$$\begin{aligned} b_2 &= \frac{(R + aY_n)^n}{(n+1)Y_n}\left(\left(\dot{R} + \dot{a}Y_n\right)\left(R^2 + 2RaY_n + a^2Y_n^2\right) - R^2\dot{R}\right) \\ &= \frac{R^n}{(n+1)Y_n}\left(2R\dot{R}aY_n + \dot{a}Y_nR^2\right) + \mathcal{O}\left(\varepsilon^2\right) \\ &= \frac{R^{n+2}}{(n+1)}\left(\dot{a} + 2a\frac{\dot{R}}{R}\right) + \mathcal{O}\left(\varepsilon^2\right). \end{aligned} \qquad \ldots(15)$$

Thus we have that upon substitution into Eq. 12, the potential

$$\phi = \begin{cases} \phi_1 = \dfrac{R^2\dot{R}}{r} - \dfrac{r^n}{nR^{n-1}} Y_n \left(\dot{a} + 2a\dfrac{\dot{R}}{R}\right), & \text{for } r < R; \\ \phi_2 = \dfrac{R^2\dot{R}}{r} + \dfrac{R^{n+2}}{(n+1)r^{n+1}} Y_n \left(\dot{a} + 2a\dfrac{\dot{R}}{R}\right), & \text{for } r > R, \end{cases} \quad \ldots(16)$$

is indeed valid at $\mathcal{O}(\varepsilon)$ which verifies (4) and (5).

Bernoulli's Equation

Plesset then proceeds to use Bernoulli's[17] equation A,

$$p = P(t) + \rho \left[\frac{\partial \phi}{\partial t} - \frac{1}{2}|\nabla\phi|^2\right], \quad \ldots(17)$$

to evaluate the pressure on either side of the interface surface. Thus, if p_1 is the pressure at the interface in region 1 and p_2 is the pressure at the interface in region 2, we have

$$p_1 = P_1(t) + \rho_1 \left[\frac{\partial \phi_1}{\partial t} - \frac{1}{2}\left(\frac{\partial \phi_1}{\partial r}\right)^2\right]_{r=r_s} \quad \ldots(18)$$

and

$$p_2 = P_2(t) + \rho_2 \left[\frac{\partial \phi_2}{\partial t} - \frac{1}{2}\left(\frac{\partial \phi_2}{\partial r}\right)^2\right]_{r=r_s}. \quad \ldots(19)$$

$P_1(t)$ and $P_2(t)$ are the constants of the spatial integration of the equation of motion which lead to the Bernoulli integral; $P_2(t)$ has the further significance of being the pressure at infinity.

Evaluate $\dfrac{\partial\phi}{\partial t}$ and $\dfrac{\partial\phi}{\partial r}$ up to $\mathcal{O}(\varepsilon)$

We next proceed to prove the quantities found in (8), (9) and (10). Taking the potential

$$\phi_1 = \frac{R^2\dot{R}}{r} - \frac{r^n}{nR^{n-1}} Y_n \left(\dot{a} + 2a\frac{\dot{R}}{R}\right),$$

we evaluate $\dfrac{\partial \phi_1}{\partial t}$ term by term.

The first term up to $\mathcal{O}(\varepsilon)$ is:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial t}\left(R^2\dot{R}\right)_{r=r_s} &= \frac{1}{R + aY_n}\frac{d}{dt}\left(R^2\dot{R}\right) \\ &= R^{-1}\left(1 + \frac{aY_n}{R}\right)^{-1}\frac{d}{dt}\left(R^2\dot{R}\right) \\ &= \frac{1}{R}\frac{d}{dt}\left(R^2\dot{R}\right) - \frac{aY_n}{R^2}\frac{d}{dt}\left(R^2\dot{R}\right) + \mathcal{O}(\varepsilon^2) \\ &= 2\dot{R}^2 + R\ddot{R} - aY_n\left(2\frac{\dot{R}^2}{R} + \ddot{R}\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad \ldots(20)$$

The second term up to $\mathcal{O}(\varepsilon)$ is:

$$\begin{aligned} -\frac{r^nY_n}{n}\frac{\partial}{\partial t}\left(\frac{\dot{a}}{R^{n-1}}\right)_{r=r_s} &= -\frac{(R + aY_n)^nY_n}{n}\left(\frac{\ddot{a}}{R^{n-1}} - (n-1)\frac{\dot{a}}{R^n}\right) \\ &= -\frac{R^nY_n}{n}\left(\frac{\ddot{a}}{R^{n-1}} - (n-1)\dot{R}\frac{\dot{a}}{R^n}\right) + \mathcal{O}(\varepsilon^2) \\ &= \frac{Y_n}{n}\left((n-1)\dot{a}\dot{R} - \ddot{a}R\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad \ldots(21)$$

The third term up to $\mathcal{O}(\varepsilon)$ is:

$$
\begin{aligned}
-\frac{2Y_n r^n}{n}\frac{\partial}{\partial t}\Big(\frac{a}{R^n}\dot{R}\Big)_{r=r_s} &= -\frac{2Y_n(R+aY_n)^n}{n}\left(\frac{\dot{a}}{R^n}\dot{R} - n\frac{a}{R^{n+1}}\dot{R}\dot{R} + \frac{a}{R^n}\ddot{R}\right) \\
&= -\frac{2Y_n R^n}{n}\left(\frac{\dot{a}}{R^n}\dot{R} - n\frac{a}{R^{n+1}}\dot{R}^2 + \frac{a}{R^n}\ddot{R}\right) + \mathcal{O}(\varepsilon^2) \\
&= \frac{2Y_n}{n}\left(na\frac{\dot{R}^2}{R} - \dot{a}\dot{R} - a\ddot{R}\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\qquad \ldots(22)
$$

Similarly, taking the potential

$$
\phi_2 = \frac{R^2\dot{R}}{r} + \frac{R^{n+2}}{(n+1)r^{n+1}}Y_n\left(\dot{a} + 2a\frac{\dot{R}}{R}\right),
$$

we evaluate $\dfrac{\partial\phi_2}{\partial t}$ term by term.

The first term up to $\mathcal{O}(\varepsilon)$ is:

$$
\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial t}\Big(R^2\dot{R}\Big)_{r=r_s} &= \frac{1}{R+aY_n}\frac{d}{dt}\Big(R^2\dot{R}\Big) \\
&= R^{-1}\left(1+\frac{aY_n}{R}\right)^{-1}\frac{d}{dt}\Big(R^2\dot{R}\Big) \\
&= \frac{1}{R}\frac{d}{dt}\Big(R^2\dot{R}\Big) - \frac{aY_n}{R^2}\frac{d}{dt}\Big(R^2\dot{R}\Big) + \mathcal{O}(\varepsilon^2) \\
&= 2\dot{R}^2 + R\ddot{R} - aY_n\left(2\frac{\dot{R}^2}{R} + \ddot{R}\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\qquad \ldots(23)
$$

The second term up to $\mathcal{O}(\varepsilon)$ is:

$$
\begin{aligned}
\frac{Y_n}{(n+1)r^{n+1}}\frac{\partial}{\partial t}\big(\dot{a}R^{n+2}\big)_{r=r_s} &= \frac{Y_n}{(n+1)(R+aY_n)^{n+1}}\Big(\ddot{a}R^{n+2} + (n+2)\dot{a}R^{n+1}\dot{R}\Big) \\
&= \frac{Y_n}{(n+1)R^{n+1}}\Big(\ddot{a}R^{n+2} + (n+2)\dot{a}R^{n+1}\dot{R}\Big) + \mathcal{O}(\varepsilon^2) \\
&= \frac{Y_n}{n+1}\Big(\ddot{a}R + (n+2)\dot{a}\dot{R}\Big) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\qquad \ldots(24)
$$

The third term up to $\mathcal{O}(\varepsilon)$ is:

$$
\begin{aligned}
\frac{2Y_n}{(n+1)r^{n+1}}\frac{\partial}{\partial t}\Big(R^{n+1}a\dot{R}\Big)_{r=r_s} &= \frac{2Y_n}{(n+1)(R+aY_n)^{n+1}}\Bigg((n+1)\dot{R}R^n a\dot{R} + \cdots \\
&\qquad \cdots + R^{n+1}\dot{a}\dot{R} + R^{n+1}a\ddot{R}\Bigg) + \mathcal{O}(\varepsilon^2) \\
&= \frac{2Y_n}{(n+1)}\left((n+1)a\frac{\dot{R}^2}{R} + \dot{a}\dot{R} + a\ddot{R}\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\qquad \ldots(25)
$$

Upon adding these three terms together for both ϕ_1 and ϕ_2, we obtain up to $\mathcal{O}(\varepsilon)$

$$
\begin{aligned}
\frac{\partial\phi_1}{\partial t}\bigg|_{r=r_s} &= \frac{1}{R}\frac{d}{dt}\Big(R^2\dot{R}\Big) - \frac{aY_n}{R^2}\frac{d}{dt}\Big(R^2\dot{R}\Big) - \frac{\ddot{a}}{n}RY_n + \cdots \\
&\cdots + \frac{n-3}{n}\dot{a}\dot{R}Y_n - \frac{2a}{n}\frac{d^2R}{dt^2}Y_n + 2a\frac{\dot{R}^2}{R}Y_n + \mathcal{O}(\varepsilon^2);
\end{aligned}
\qquad \ldots(26)
$$

$$
\begin{aligned}
\frac{\partial\phi_2}{\partial t}\bigg|_{r=r_s} &= \frac{1}{R}\frac{d}{dt}\Big(R^2\dot{R}\Big) - \frac{aY_n}{R^2}\frac{d}{dt}\Big(R^2\dot{R}\Big) + \frac{\ddot{a}}{n+1}RY_n + \cdots \\
&\cdots + \frac{n+4}{n+1}\dot{a}\dot{R}Y_n + \frac{2a}{n+1}\frac{d^2R}{dt^2}Y_n + 2a\frac{\dot{R}^2}{R}Y_n + \mathcal{O}(\varepsilon^2);
\end{aligned}
\qquad \ldots(27)
$$

$$\left(\frac{\partial\phi_1}{\partial r}\right)^2\Bigg|_{r=r_s} \simeq \left(\frac{\partial\phi_2}{\partial r}\right)^2\Bigg|_{r=r_s} \simeq \dot{R}^2 + 2\dot{a}\dot{R}Y_n + \mathcal{O}(\varepsilon^2). \qquad \ldots(28)$$

These are the quantities found in (8), (9) and (10). Simplified, we have

$$\begin{aligned}
\frac{\partial\phi_1}{\partial t}\Bigg|_{r=r_s} &= R\ddot{R} + 2\dot{R}^2 + \frac{Y_n}{n}\Big((n-3)\dot{a}\dot{R} - (n+2)a\ddot{R} - \ddot{a}R\Big) + \mathcal{O}(\varepsilon^2);\\
\frac{\partial\phi_2}{\partial t}\Bigg|_{r=r_s} &= R\ddot{R} + 2\dot{R}^2 + \frac{Y_n}{n+1}\Big((n+4)\dot{a}\dot{R} + \ddot{a}R - (n-1)a\ddot{R}\Big) + \mathcal{O}(\varepsilon^2);\\
\left(\frac{\partial\phi_1}{\partial r}\right)^2\Bigg|_{r=r_s} &\simeq \left(\frac{\partial\phi_2}{\partial r}\right)^2\Bigg|_{r=r_s} \simeq \dot{R}^2 + 2\dot{a}\dot{R}Y_n + \mathcal{O}(\varepsilon^2).
\end{aligned} \qquad \ldots(29)$$

It may be noted that while the components of velocity perpendicular to the radial velocity are of first order, their contributions to $\left(\frac{\partial\phi}{\partial r}\right)^2_{r_s}$ are of second order and are therefore to be neglected.

Equation of motion up to $\mathcal{O}(\varepsilon)$

The pressures at the interface are connected by the relation

$$p_2 = p_1 - \sigma\left(\frac{1}{R'} + \frac{1}{R''}\right),$$

where R' and R'' are the principle radii of curvature of the interface and σ is the surface tension. To $\mathcal{O}(\varepsilon)$ Lamb[16] obtains

$$\frac{1}{R'} + \frac{1}{R''} = \frac{2}{R} + \frac{(n-1)(n+2)}{R^2}aY_n + \mathcal{O}(\varepsilon^2),$$

so that

$$p_2 = p_1 - \frac{2\sigma}{R} - \frac{(n-1)(n+2)}{R^2}\sigma a Y_n + \mathcal{O}(\varepsilon^2). \qquad \ldots(30)$$

We proceed further, and follow Plesset's line of reasoning from Eq. 26, 27 and 28. The terms in this relation between p_2 and p_1 which are independent of Y_n give the equation of motion for the unperturbed interface now known as the *Rayleigh-Plesset Equation*:

$$\begin{aligned}
P_2 + \rho_2\left[\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) - \frac{1}{2}\dot{R}^2\right] &= P_1 + \rho_1\left[\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) - \frac{1}{2}\dot{R}^2\right] - \frac{2\sigma}{R}\\
2\dot{R}^2 + R\ddot{R} - \frac{1}{2}\dot{R}^2 &= \frac{P_1 - P_2 - \frac{2\sigma}{R}}{\rho_2 - \rho_1}\\
R\ddot{R} + \frac{3}{2}\dot{R}^2 &= \frac{P_1 - P_2 - \frac{2\sigma}{R}}{\rho_2 - \rho_1}.
\end{aligned} \qquad \ldots(31)$$

The terms proportional to Y_n in Eq. 30 give

$$\begin{aligned}
&\rho_2\left[-\frac{a}{R^2}\frac{d}{dt}(R^2\dot{R}) + \frac{\ddot{a}}{n+1}R + \frac{n+4}{n+1}\dot{a}\dot{R} + \frac{2a}{n+1}\frac{d^2R}{dt^2} + 2a\frac{\dot{R}^2}{R} - \dot{a}\dot{R}\right] = \cdots\\
&\cdots = \rho_1\left[-\frac{a}{R^2}\frac{d}{dt}(R^2\dot{R}) - \frac{\ddot{a}}{n}R + \frac{n-3}{n}\dot{a}\dot{R} - \frac{2a}{n}\frac{d^2R}{dt^2} + 2a\frac{\dot{R}^2}{R} - \dot{a}\dot{R}\right] - \cdots\\
&\cdots - \frac{(n-1)(n+2)}{R^2}\sigma a.
\end{aligned} \qquad \ldots(32)$$

Rearranging we have

$$\begin{aligned}&\ddot{a}\left(\frac{R}{n+1}\rho_2+\frac{R}{n}\rho_1\right)+\dot{a}\left(\frac{n+4}{n+1}\dot{R}\rho_2-\frac{n-3}{n}\dot{R}\rho_1-(\rho_2-\rho_1)\dot{R}\right)+\cdots\\&\cdots+a\left(\frac{2}{n+1}\ddot{R}\rho_2+\frac{2}{n}\ddot{R}\rho_1+(\rho_2-\rho_1)\left(2\frac{\dot{R}^2}{R}\right)-(\rho_2-\rho_1)\left(2\frac{\dot{R}^2}{R}+\ddot{R}\right)+\cdots\right.\\&\left.\cdots+\frac{\sigma(n-1)(n+2)}{R^2}\right)=0.\end{aligned} \qquad \ldots(33)$$

Combining the fractions gives us

$$\begin{aligned}&\ddot{a}\left(\frac{n\rho_2+\rho_1(n+1)}{n(n+1)}R\right)+\dot{a}\left(\frac{n(n+4)\rho_2-(n-3)(n+1)\rho_1-n(n+1)(\rho_2-\rho_1)}{n(n+1)}\dot{R}\right)+\cdots\\&\cdots+a\left(\frac{2n\rho_2+2(n+1)\rho_1-n(n+1)(\rho_2-\rho_1)}{n(n+1)}\ddot{R}+\frac{\sigma(n-1)(n+2)}{R^2}\right)=0.\end{aligned} \qquad \ldots(34)$$

Further simplifying the numerators, it follows that

$$\begin{aligned}&\ddot{a}\left(\frac{n\rho_2+(n+1)\rho_1}{n(n+1)}R\right)+\dot{a}\left(\frac{3(n\rho_2+(n+1)\rho_1)}{n(n+1)}\dot{R}\right)+\cdots\\&\cdots+a\left(\frac{n(1-n)\rho_2+(n+1)(n+2)\rho_1}{n(n+1)}\ddot{R}+\frac{\sigma(n-1)(n+2)}{R^2}\right)=0.\end{aligned} \qquad \ldots(35)$$

Final Result

Dividing through by $\dfrac{n\rho_2+(n+1)\rho_1}{n(n+1)}R$, we finally arrive at

$$\ddot{a}+\frac{3\dot{R}}{R}\dot{a}-A_n a=0,$$

where

$$\begin{aligned}A_n&=\frac{\ddot{R}\big(n(n-1)\rho_2-(n+1)(n+2)\rho_1\big)}{R\big(n\rho_2+(n+1)\rho_1\big)}-\omega_n^2;\\\omega_n^2&=\frac{n(n-1)(n+1)(n+2)\sigma}{\big(n\rho_2+(n+1)\rho_1\big)R^3}.\end{aligned}$$

This is the differential equation for a from which stability conditions may be deduced, since coined *Bell-Plesset Instability*,[5,8] and proves Plesset's result in (13) and (14).

III My Approach to Include Viscosity

Bernoulli's equation including viscosity, assuming zero vorticity from (A), is

$$P_1(t) + \rho_1 \left[\frac{\partial \phi_1}{\partial t} - \frac{1}{2}\left(\frac{\partial \phi_1}{\partial r}\right)^2 - \frac{\nu_1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_1}{\partial r}\right) - 2\phi_1\right)\right]_{r=r_s} = \cdots$$
$$\cdots = P_2(t) + \rho_2 \left[\frac{\partial \phi_2}{\partial t} - \frac{1}{2}\left(\frac{\partial \phi_2}{\partial r}\right)^2 - \frac{\nu_2}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_2}{\partial r}\right) - 2\phi_2\right)\right]_{r=r_s}. \quad \ldots(36)$$

Thus we need to evaluate

$$\left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_1}{\partial r}\right) - 2\phi_1\right)\right]_{r=r_s} = -\frac{2\dot{R}}{R} - \frac{Y_n}{nR^2}\Big((n-1)(n+2)\dot{a}R + \cdots$$
$$\cdots + 2\left(n^2 - 2n - 2\right) a\dot{R}\Big) + \mathcal{O}\left(\varepsilon^2\right);$$
$$\left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_2}{\partial r}\right) - 2\phi_2\right)\right]_{r=r_s} = -\frac{2\dot{R}}{R} + \frac{Y_n}{(n+1)R^2}\Big((n-1)(n+2)\dot{a}R + \cdots$$
$$\cdots + 2\left(n^2 + 4n + 1\right) a\dot{R}\Big) + \mathcal{O}\left(\varepsilon^2\right). \quad \ldots(37)$$

We substitute the values obtained previously independent of Y_n to arrive at

$$P_1 + \rho_1\left[\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) - \frac{1}{2}\dot{R}^2 + \nu_1\left(\frac{2\dot{R}}{R}\right)\right] = \cdots$$
$$\cdots = P_2 + \rho_2\left[\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) - \frac{1}{2}\dot{R}^2 + \nu_2\left(\frac{2\dot{R}}{R}\right)\right] - \frac{2\sigma}{R}. \quad \ldots(38)$$

This gives us the correct *Rayleigh-Plesset* equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{P_2 - P_1}{\rho_2 - \rho_1} + \frac{2\sigma}{(\rho_2 - \rho_1)R} + \frac{2\dot{R}}{R}\left(\frac{\mu_2 - \mu_1}{\rho_2 - \rho_1}\right) = 0, \quad \ldots(39)$$

suggesting that the discrepancy between the two approaches only occurs at higher orders.

IV Comparison Between Plesset and Amendt

Drawing comparisons in the case of vanishing surface tension and viscosity, we have from both treatments of Plesset [4–7] and Amendt [13] respectively,

$$\ddot{a} + \dot{a}\frac{3\dot{R}}{R} + a\left(\frac{\ddot{R}}{R}\mathbb{A}_3 + \frac{\mathbb{S}_1}{R^3}\right) = 0,$$
$$\ddot{a} + \dot{a}\frac{3\dot{R}}{R}\mathbb{A}_2 + a\left(\frac{\ddot{R}}{R}\mathbb{A}_1 + \frac{\mathbb{S}_1}{R^3}\right) = 0, \quad \ldots(40)$$

where

$$\mathbb{A}_3 = \frac{(n+1)(n+2)\rho_1 - n(n-1)\rho_2}{n\rho_2 + (n+1)\rho_1}. \quad \ldots(41)$$

The term responsible for *Bell-Plesset* instability is $\dot{a}$. For the case of a bubble, we can neglect ρ_1, where in the following as $\rho_1 \to 0$

$$\begin{aligned} &\mathbb{A}_1 = -(n-1); && \\ &\mathbb{A}_2 = 1; && \mathbb{V}_1 = 0; \\ &\mathbb{A}_3 = -(n-1); && \mathbb{V}_2 = 0; \\ &\mathbb{S}_1 = \frac{(n-1)(n+1)(n+2)\sigma}{\rho_2 R^3}; && \mathbb{V}_3 = 0. \end{aligned} \quad \ldots(42)$$

This gives us, neglecting $\mathbb{S}$,

$$\ddot{a} + \dot{a}\frac{3\dot{R}}{R} - a(n-1)\frac{\ddot{R}}{R} = 0, \quad \ldots(43)$$

for both velocity potentials. However, for a liquid droplet with a surrounding gas, contrasting behaviour arises. To see this, as $\rho_2 \to 0$

$$\begin{aligned} &\mathbb{A}_1 = (n-1); && \\ &\mathbb{A}_2 = 0; && \mathbb{V}_1 = 0; \\ &\mathbb{A}_3 = (n+2); && \mathbb{V}_2 = 0; \\ &\mathbb{S}_1 = \frac{n(n-1)(n+2)\sigma}{\rho_1 R^3}; && \mathbb{V}_3 = 0. \end{aligned} \quad \ldots(44)$$

Neglecting $\mathbb{S}$, Plesset's reduces to

$$\ddot{a} + \dot{a}\frac{3\dot{R}}{R} + a(n+2)\frac{\ddot{R}}{R} = 0, \quad \ldots(45)$$

and Amendt's reduces to

$$\ddot{a} + a(n-1)\frac{\ddot{R}}{R} = 0. \quad \ldots(46)$$

Plesset made the claim that an instability exists even in the Rayleigh-Taylor stable case when $\ddot{R} > 0$, provided that

$$(2n+1)R\ddot{R} < \frac{3\dot{R}^2}{2}. \quad \ldots(47)$$

Contrary to this, (46) shows that the middle term responsible for growth in the limit of large ρ_1 vanishes.

We can therefore conclude that an expanding $\left(\dot{R} > 0\right)$ and accelerating $\left(\ddot{R} > 0\right)$, high density ($\rho_1 \gg \rho_2$) bubble, is not unstable within the limits of this perturbation, and in this limit, compressibility need not be considered.

V Stability Conditions

To begin solving the equation analytically we can make the substitution

$$a = \alpha \exp\left[-\frac{3}{2}\int_{t_0}^{t}\frac{\dot{R}(t')}{R(t')}dt'\right] = \alpha\left(\frac{R_0}{R}\right)^{\frac{3}{2}}, \qquad \ldots(48)$$

where it follows that

$$\begin{aligned}\dot{a} &= \left(\frac{R_0}{R}\right)^{\frac{3}{2}}\left(\dot{\alpha} - \frac{3\dot{R}}{2R}\alpha\right);\\ \ddot{a} &= \left(\frac{R_0}{R}\right)^{\frac{3}{2}}\left(\ddot{\alpha} - \frac{3\dot{R}}{R}\dot{\alpha} + \alpha\left(\left(\frac{9}{4}+\frac{3}{2}\right)\frac{\dot{R}^2}{R} - \frac{3\ddot{R}}{2R}\right)\right).\end{aligned} \qquad \ldots(49)$$

This gives us the new relationship

$$\ddot{\alpha} + Q(t)\alpha = 0, \qquad \ldots(50)$$

where, including surface tension,

$$Q(t) = \frac{\mathbb{S}_1}{R^3} + \frac{\dot{R}^2}{R^2}\mathbb{A}_2\left(\frac{3}{2} - \frac{9}{4}\mathbb{A}_2\right) + \frac{\ddot{R}}{R}\left(\mathbb{A}_1 - \frac{3}{2}\mathbb{A}_2\right). \qquad \ldots(51)$$

In terms of stability[1] conditions, we have stability when $Q(t) < 0$, and instability when $Q(t) > 0$.

Case 1: $\rho_2 \gg \rho_1$

The function $Q(t)$ simplifies to

$$Q(t) = \frac{(n-1)(n+1)(n+2)\sigma}{\rho_2 R^3} - \frac{3\dot{R}^2}{4R^2} - \frac{\ddot{R}}{R}\left(n + \frac{1}{2}\right). \qquad \ldots(52)$$

Case 2: $\rho_1 \gg \rho_2$

The function $Q(t)$ simplifies to

$$Q(t) = \frac{n(n-1)(n+2)\sigma}{\rho_1 R^3} + \frac{\ddot{R}}{R}(n-1). \qquad \ldots(53)$$

VI Solution (Incompressible)

Collapsing Gas

The solution to (50) is obtained via a *WKBJ* approximation, outlined in (C), which gives us

$$a \simeq a_0 R^{-\frac{1}{4}}\cos\left(\lambda c\int^{t} R^{-\frac{5}{2}}\mathrm{d}t'\right), \quad R \to 0, \qquad \ldots(54)$$

with the boundary conditions applied $a(0) = a_0$ and $\dot{a}(0) = \dot{a}_0$ respectively. Fig. (2) illustrates the dominant proportionality of the amplitude $a \propto R^{-\frac{1}{4}}$ as $R \to 0$.

[1]Stability (or instability) in this case refers to an exponential decrease (or increase) in the amplitude of the disturbance.

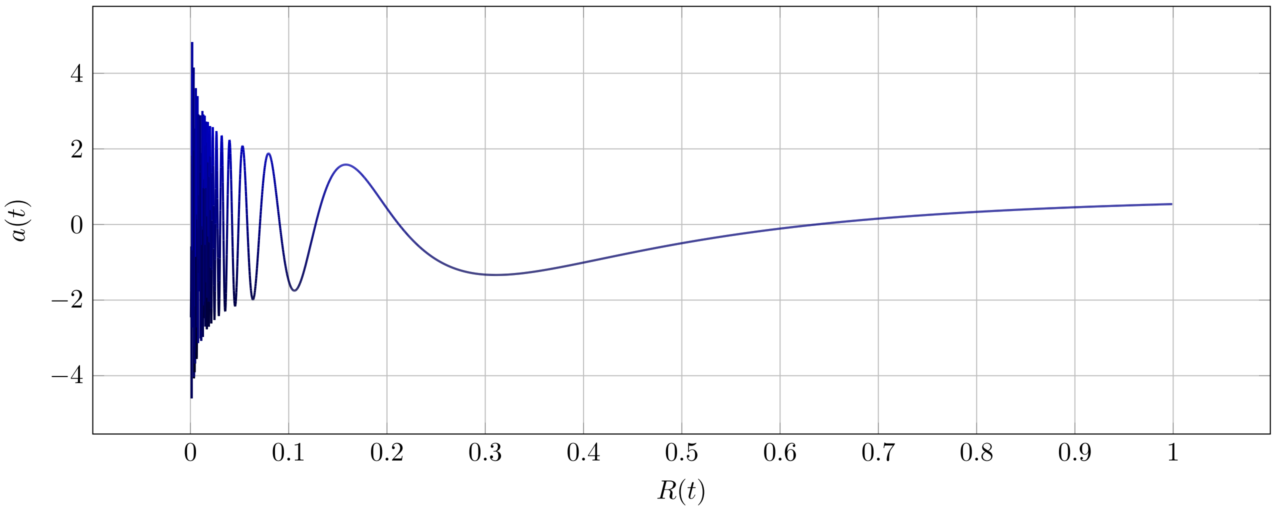

Figure 2: $a \propto R^{-\frac{1}{4}} \cos\left(\lambda c \int^t R^{-\frac{5}{2}} \mathrm{d}t'\right)$ and oscillates with increasing frequency as $R \to 0$. At approximately $R = 2$ onwards, the amplitude $a \to 0$.

VII Future Investigation

The approximate solution obtained in (VI) offers us insight into the unstable behaviour during the collapse, visualised in Figure (2). As expected, a large bubble would be influenced by surface tension effects, tending towards stability. Conversely, a smaller bubble would experience chaotic instability as it collapses.

Spherical compressibility has been shown to stabilise the oscillations of a collapsing bubble,[13] thus in future analysis I feel the incompressible case (55) to show the most merit. Potentially the compressible terms can be neglected in the region we propose within the perturbation of the interface.

Consideration to viscous effects in an incompressible regime appears to be the best direction to proceed in. The viscous terms here show the same proportionality to those approximated in Prosperitti's[11] work, suggesting close validity in agreement. As was stated before, Prosperitti's work agrees with the results for the frequency of oscillation ω_0 and the decay constant b_0 obtained by Lamb,[16] in contrast to the Atwood numbers $\mathbb{V}_1$ and $\mathbb{V}_2$ obtained in my own approach.

The main progress made is verification of Plesset[5], Amendt[13], Fisher[10] and Bell's[8] results, of which a determination of the correct potential to use in the specific case of a collapsing bubble was found. A discrepancy in agreement between the viscous correction of Prosperitti's and my own was discovered; thus future verification of Prosperitti's result is the next course of action. Once this has been achieved, direct comparison can be made, and plans to solve both numerically to compare against real lab data can help to assess validity of both claims.

Chandrasekar[18] also performed an analysis in a similar vein as Prosperitti, and I plan to review this approach and determine its adaptability to the derivation obtained here. The core of his work involved the mechanics of spheres, and their application to astrophysical problems. Sharing this interest, I hope to determine feasibility of this analysis towards the surface of a star, and the following consequences of the collapse prior to supernovae.[19] Little investigation seems to have been carried out concerning the stability of the photosphere and chromosphere under collapse. Should this exceed the scope of my initial investigation too far however, it may be best to consider the base mechanics at play and leave this idea for future work.

Incompressible Bell-Plesset Instability with Viscous Terms and Surface Tension

$$\ddot{a} + \dot{a}\left(\frac{3\dot{R}}{R}\mathbb{A}_2 + \frac{\mathbb{V}_1}{R^2}\right) + a\left(\frac{\dot{R}\mathbb{V}_2}{R^3} + \frac{\ddot{R}\mathbb{A}_1}{R} + \frac{\mathbb{S}_1}{R^3}\right) = 0. \qquad \ldots(55)$$

A Proof of Bernoulli's Compressible Equation

Here we demonstrate Bernoulli's equation incorporating compressibility and viscosity.[17,20] The momentum equation is

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=\nabla\cdot\underline{\underline{\boldsymbol{\sigma}}}+\rho\mathbf{g}, \qquad \ldots(56)$$

where

$$\underline{\underline{\boldsymbol{\sigma}}}=-p\underline{\underline{\mathbf{I}}}+2\mu\Big(\underline{\underline{\mathbf{e}}}+\underline{\underline{\boldsymbol{\Omega}}}\Big)+\lambda(\nabla\cdot\mathbf{u})\underline{\underline{\mathbf{I}}}, \qquad \ldots(57)$$

and

$$\underline{\underline{\mathbf{e}}}=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}}\right), \qquad \underline{\underline{\boldsymbol{\Omega}}}=\frac{1}{2}\left(\nabla\mathbf{u}-\nabla\mathbf{u}^{\mathrm{T}}\right). \qquad \ldots(58)$$

Using the vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u}=\frac{1}{2}\nabla\left|\mathbf{u}\right|^2-\mathbf{u}\times(\nabla\times\mathbf{u}) \qquad \ldots(59)$$

with vorticity $\boldsymbol{\omega}=\nabla\times\mathbf{u}$, we have

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{2}\nabla\left|\mathbf{u}\right|^2\right]-\nabla\cdot\underline{\underline{\boldsymbol{\sigma}}}-\rho\mathbf{g}=\mathbf{u}\times\boldsymbol{\omega}. \qquad \ldots(60)$$

If all body forces are conservative, then $\mathbf{g}=-\nabla\Phi$. For a Compressible, Newtonian fluid with dynamic viscosity μ,

$$\begin{aligned}\nabla\cdot\underline{\underline{\boldsymbol{\sigma}}}&=\nabla\cdot\left[-p\underline{\underline{\mathbf{I}}}+\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}}\right)+\lambda(\nabla\cdot\mathbf{u})\underline{\underline{\mathbf{I}}}\right]\\&=\frac{\partial}{\partial x_j}\left[-p\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\\&=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\mu\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}+\lambda\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k}\\&=-\nabla p+\mu\nabla^2\mathbf{u}+\mu\nabla(\nabla\cdot\mathbf{u})+\lambda\nabla(\nabla\cdot\mathbf{u}).\end{aligned} \qquad \ldots(61)$$

Inserting (61) into (60) while neglecting both $\underline{\underline{\boldsymbol{\Omega}}}$ and $\lambda=-\frac{2}{3}\mu$, gives us

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{2}\nabla\left|\mathbf{u}\right|^2\right]+\nabla p-\mu\nabla^2\mathbf{u}-\mu\nabla(\nabla\cdot\mathbf{u})+\rho\nabla\Phi=\mathbf{u}\times\boldsymbol{\omega}, \qquad \ldots(62)$$

where $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity. If the flow is irrotational, then $\boldsymbol{\omega}=\mathbf{0}$ and $\mathbf{u}=-\nabla\phi$ using Helmholtz decomposition. Substituting, we have

$$\rho\left[-\frac{\partial(\nabla\phi)}{\partial t}+\frac{1}{2}\nabla\left|\nabla\phi\right|^2\right]+\nabla p+\mu\nabla^2(\nabla\phi)+\mu\nabla(\nabla\cdot(\nabla\phi))+\rho\nabla\Phi=\mathbf{0}. \qquad \ldots(63)$$

All operations are linear, so can be interchanged. We then integrate to arrive at the relation

$$\begin{aligned}P(t)&=\rho\left[-\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2\right]+p+\mu\nabla^2\phi+\mu\nabla^2\phi+\rho\Phi,\\\Leftrightarrow P(t)&=-\rho\left[\frac{\partial\phi}{\partial t}-\frac{1}{2}\left|\nabla\phi\right|^2\right]+p+2\mu\nabla^2\phi+\rho\Phi.\end{aligned} \qquad \ldots(64)$$

This becomes Bernoulli's equation including viscous terms

$$p=P(t)-2\mu\nabla^2\phi+\rho\left[\frac{\partial\phi}{\partial t}-\frac{1}{2}\left|\nabla\phi\right|^2-\Phi\right]. \qquad \ldots(65)$$

B Viscous Terms

The following is the solution to the laplacian of ϕ, up to order $\mathcal{O}(\varepsilon)$.

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi_1}{\partial r}\right)-\frac{2\phi_1}{r^2}\right]_{r=r_s}=-\frac{2\dot{R}}{R}-\frac{(n+2)(n-1)Y_n}{nR^2}\left(\dot{a}R-a\dot{R}\right)+\mathcal{O}(\varepsilon^2);$$
$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi_2}{\partial r}\right)-\frac{2\phi_2}{r^2}\right]_{r=r_s}=-\frac{2\dot{R}}{R}+\frac{Y_n}{(n+1)R^2}\Big(n(n+3)aF_2R+\cdots \qquad \ldots(66)$$
$$\cdots+2\left(n^2+4n+1\right)a\dot{R}+(n+2)(n-1)\dot{a}R\Big)+\mathcal{O}(\varepsilon^2).$$

C WKBJ Approximation

C.1 Collapsing Gas (Incompressible)

By the *WKBJ* approximation we can solve for the case of a collapsing bubble,

$$\ddot{\alpha}+Q(t)\alpha=0, \quad \text{where} \quad Q(t)=\frac{(n-1)(n+1)(n+2)\sigma}{\rho_2R^3}-\frac{3\dot{R}^2}{4R^2}-\frac{\ddot{R}}{R}\left(n+\frac{1}{2}\right). \qquad \ldots(67)$$

The radius of the unperturbed bubble is given by

$$R\ddot{R}+\frac{3}{2}\dot{R}^2+\frac{P_2(t)-P_1(t)}{\rho_2}+\frac{2\sigma}{\rho_2R}=0. \qquad \ldots(68)$$

where $P_1(t)$ is the pressure inside the bubble, and $P_2(t)$ is the pressure a distance from the bubble. We may also write this as

$$\frac{d}{dt}\left(R^3\dot{R}^2\right)=2R^2\dot{R}\left(\frac{P_1(t)-P_2(t)}{\rho_2}-\frac{2\sigma}{\rho_2R}\right) \qquad \ldots(69)$$

which integrates, when $P_1(t)-P_2(t)$ is a constant, to give

$$2\pi\rho_2\left(R^3\dot{R}^2-R_0^3\dot{R}_0^2\right)=\frac{4\pi}{3}(P_1(t)-P_2(t))\left(R^3-R_0^3\right)-4\pi\sigma\left(R^2-R_0^2\right) \qquad \ldots(70)$$

where R_0 is the cavity radius and $\dot{R}$ is its radial velocity at time $t=t_0$. From (70) we have

$$\dot{R}^2=\mathcal{O}\left(R^{-1}\right)+\frac{R_0^3}{R^3}\left(\dot{R}_0^2+\frac{2p}{3\rho_2}+\frac{2\sigma}{\rho_2R_0}\right), \qquad \ldots(71)$$

where $p=P_2(t)-P_1(t)>0$ in this case. The radial acceleration, $\ddot{R}$ is determined from (69)

$$\ddot{R}=-\frac{3\dot{R}^2}{2R}+\frac{p}{\rho_2R}-\frac{2\sigma}{\rho_2R^2}. \qquad \ldots(72)$$

Substituting (71) and (72) into (67) and neglecting surface tension,

$$Q(t)=-\frac{3R_0^3}{4R^5}\left(\dot{R}_0^2+\frac{2p}{3\rho_2}+\frac{2\sigma}{\rho_2R_0}\right)-\frac{\left(n+\frac{1}{2}\right)}{R}\left(-\frac{3R_0^3}{2R^4}\left(\dot{R}_0^2+\frac{2p}{3\rho_2}+\frac{2\sigma}{\rho_2R_0}\right)+\frac{p}{\rho_2R}-\frac{2\sigma}{\rho_2R^2}\right). \qquad \ldots(73)$$

The function $Q(t)$ then becomes

$$Q(t)\simeq\frac{3nR_0^3}{2R^5}\left(\dot{R}_0^2+\frac{2p}{3\rho_2}+\frac{2\sigma}{\rho_2R_0}\right), \quad R\to 0, \qquad \ldots(74)$$

neglecting terms smaller than $\mathcal{O}\left(R^{-5}\right)$, and simplifies to

$$Q(t)\simeq\frac{nc^2}{R^5}, \qquad \ldots(75)$$

where c contains the constant terms. We can thus proceed with the *WKBJ* approximation for $\lambda \gg 1$ where we note that $Q(t) > 0$ in (75). The procedure is constructed via

$$\begin{aligned} &\ddot{\alpha} + \lambda^2 Q(t)\alpha = 0, \quad H = H_0 + \lambda^{-1}H_1 + \mathcal{O}\big(\lambda^{-2}\big), \\ &\alpha = e^{\lambda H}, \quad \dot{\alpha} = \lambda \dot{H} e^{\lambda H}, \quad \ddot{\alpha} = \left(\lambda \ddot{H} + \lambda^2 \dot{H}^2\right) e^{\lambda H}. \end{aligned} \qquad \ldots (76)$$

At $\mathcal{O}(1)$

$$\begin{aligned} \dot{H}_0^2 + Q(t) = 0 \quad \Rightarrow \quad \dot{H}_0 &= \pm i\sqrt{Q(t)} \\ H_0 &= \pm i \int^t \sqrt{Q(t')}\mathrm{d}t' + k_0. \end{aligned} \qquad \ldots (77)$$

At $\mathcal{O}\big(\lambda^{-1}\big)$

$$\begin{aligned} 2\dot{H}_0\dot{H}_1 + \ddot{H}_0 = 0 \quad \Rightarrow \quad \dot{H}_1 &= -\frac{1}{2}\frac{\ddot{H}_0}{\dot{H}_0} = -\frac{1}{2}\frac{d}{dt}\left(\ln\dot{H}_0\right) \\ H_1 &= -\frac{1}{2}\ln\dot{H}_0 + k_1 \\ H_1 &= -\frac{1}{4}\ln Q(t) + k_1 \end{aligned} \qquad \ldots (78)$$

Therefore the function H becomes

$$\begin{aligned} H &= \pm i \int^t \sqrt{Q(t')}\mathrm{d}t' - \frac{1}{4}\lambda^{-1}\ln Q(t) + k, \\ \alpha &= e^{\lambda H} = \exp\left\{ \pm i\lambda \int^t \sqrt{Q(t')}\mathrm{d}t' - \frac{1}{4}\ln Q(t) + k \right\} \\ &\simeq k \times \exp\left\{ -\frac{1}{4}\ln Q(t) \right\} \exp\left\{ \pm i\lambda \int^t \sqrt{Q(t')}\mathrm{d}t' \right\} \\ &\simeq k \times Q(t)^{-\frac{1}{4}} \exp\left\{ \pm i\lambda \int^t \sqrt{Q(t')}\mathrm{d}t' \right\}. \end{aligned} \qquad \ldots (79)$$

The general solution is then

$$\begin{aligned} \alpha &\simeq kQ(t)^{-\frac{1}{4}} \exp\left\{ \pm i\lambda \int^t \sqrt{Q(t')}\mathrm{d}t' \right\} \simeq kn^{\frac{1}{4}}c^{\frac{1}{2}}R^{\frac{5}{4}} \exp\left\{ \pm i\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right\} \\ &\simeq n^{\frac{1}{4}}c^{\frac{1}{2}}R^{\frac{5}{4}} \left\{ A\sin\left(i\lambda c n^{\frac{1}{2}} \int^t \sqrt{Q(t')}\mathrm{d}t' \right) + B\cos\left(i\lambda c n^{\frac{1}{2}} \int^t \sqrt{Q(t')}\mathrm{d}t' \right) \right\}, \quad R \to 0. \end{aligned} \qquad \ldots (80)$$

The distortion amplitude is then given by

$$\begin{aligned} a &\simeq kn^{\frac{1}{4}}c^{\frac{1}{2}}R^{-\frac{1}{4}} \exp\left\{ \pm i\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right\}, \quad R \to 0 \\ &\simeq n^{\frac{1}{4}}c^{\frac{1}{2}}R^{-\frac{1}{4}} \left\{ A\sin\left(\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right) + B\cos\left(\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right) \right\}, \quad R \to 0. \end{aligned} \qquad \ldots (81)$$

If we impose the boundary conditions of a small amplitude at time $t = 0$, $a(0) = a_0$ we find that $B = a_0$. Initially the amplitude velocity is reasonably satisfied by $\dot{a}(0) = \dot{a}_0$, which gives us $A = \frac{\dot{a}_0}{\lambda}R^{-\frac{1}{4}}$. The particular solution in this case is given by

$$\begin{aligned} a &\simeq a_0 R^{-\frac{1}{4}} \cos\left(\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right) + \frac{\dot{a}_0}{\lambda}\sin\left(\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right), \quad R \to 0 \\ &\simeq a_0 R^{-\frac{1}{4}} \cos\left(\lambda c n^{\frac{1}{2}} \int^t R^{-\frac{5}{2}}\mathrm{d}t' \right), \quad R \to 0. \end{aligned} \qquad \ldots (82)$$

D Mathematica Code

D.1 M. S. Plesset (1954)

```
In[826]:=
Format[D[f_[t], {t, n_ /; n < 3}]] := OverDot[f, n]
Format[R[t]] := R
Format[a[t]] := a
r_s := R[t] + a[t] Y_n;
ϕ1[t] := R[t]^2 R'[t]/r - (r^n Y_n)/(n R[t]^(n-1)) (a'[t] + (2 a[t] R'[t])/R[t]);
ϕ2[t] := R[t]^2 R'[t]/r + (R[t]^(n+2) Y_n)/((n + 1) r^(n+1)) (a'[t] + (2 a[t] R'[t])/R[t]);
Eval1 := Simplify[Series[∂_t ϕ1[t] /. r → r_s, {Y_n, 0, 1}]]
Eval2 := Simplify[Series[∂_t ϕ2[t] /. r → r_s, {Y_n, 0, 1}]]
Eval3 := Simplify[Series[∂_r ϕ1[t] /. r → r_s, {Y_n, 0, 1}]]
Step1 := Collect[P2 + ρ2 (Eval2 - 1/2 (Eval3)^2) -
           (P1 + ρ1 (Eval1 - 1/2 (Eval3)^2)),
           {a''[t], a'[t], a[t]}, Together]
Step2 := Collect[(n (n + 1))/(R (ρ1 + n ρ1 + n ρ2) Y_n) Step1,
           {a''[t], a'[t], a[t]}, Together]
Final0 = FullSimplify[Collect[1/2 (-2 P1 + 2 P2 - 3 ρ1 Ṙ^2 +
           3 ρ2 Ṙ^2 - 2 R ρ1 R̈ + 2 R ρ2 R̈), {R̈, Ṙ, R}]]
Final1 = FullSimplify[(3 ȧ Ṙ)/R + (R ä)/R +
           (a (2 ρ1 + 3 n ρ1 + n^2 ρ1 + n ρ2 - n^2 ρ2) R̈)/(R (ρ1 + n ρ1 + n ρ2))]
```

Out[837]= $-P_1 + P_2 - \frac{1}{2}\left(3\dot{R}^2 + 2R\ddot{R}\right)(\rho_1 - \rho_2)$

Out[838]= $\ddot{a} + \frac{3\dot{a}\dot{R}}{R} + \frac{a\ddot{R}\left((1+n)(2+n)\rho_1 - (-1+n)n\rho_2\right)}{(1+n)R\rho_1 + nR\rho_2}$

D.2 G. I. Bell (1951) and H. N. Fisher (1982)

In[811]:=

$$\mathrm{Format}[D[f\_[t],\ \{t,\ n\_\ /;\ n<3\}]] := \mathrm{OverDot}[f,\ n]$$

$$\mathrm{Format}[R[t]] := R$$

$$\mathrm{Format}[a[t]] := a$$

$$\mathrm{Format}[F_1[t]] := F_1$$

$$\mathrm{Format}[F_2[t]] := F_2$$

$$r_s := R[t] + a[t]\, Y_n;$$

$$\phi_1[t] := \frac{R[t]^2\, R'[t]}{r} - \left(\frac{R[t]^3}{3\, r} + \frac{r^2}{6}\right) F_1[t] + -\frac{r^n\, Y_n}{n\, R[t]^{n-1}} \left(a'[t] + \frac{2\, a[t]\, R'[t]}{R[t]} - a[t]\, F_1[t]\right);$$

$$\phi_2[t] := \frac{R[t]^2\, R'[t]}{r} - \left(\frac{R[t]^3}{3\, r} + \frac{r^2}{6}\right) F_2[t] + \frac{R[t]^{n+2}\, Y_n}{(n+1)\, r^{n+1}} \left(a'[t] + \frac{2\, a[t]\, R'[t]}{R[t]} - a[t]\, F_2[t]\right);$$

$$\mathrm{Eval1} := \mathrm{Simplify}[\mathrm{Series}[\partial_t \phi_1[t]\ /.\ r \to r_s,\ \{Y_n,\ 0,\ 1\}]]$$

$$\mathrm{Eval2} := \mathrm{Simplify}[\mathrm{Series}[\partial_t \phi_2[t]\ /.\ r \to r_s,\ \{Y_n,\ 0,\ 1\}]]$$

$$\mathrm{Eval3} := \mathrm{Simplify}[\mathrm{Series}[\partial_r \phi_1[t]\ /.\ r \to r_s,\ \{Y_n,\ 0,\ 1\}]]$$

$$\mathrm{Step1} := \mathrm{Collect}\left[P_2 + \rho_2 \left(\mathrm{Eval2} - \frac{1}{2} (\mathrm{Eval3})^2\right) - \left(P_1 + \rho_1 \left(\mathrm{Eval1} - \frac{1}{2} (\mathrm{Eval3})^2\right)\right),\ \{a''[t],\ a'[t],\ a[t]\},\ \mathrm{Together}\right]$$

$$\mathrm{Step2} := \mathrm{Collect}\left[\frac{n\,(n+1)}{R\,(\rho_1 + n\,\rho_1 + n\,\rho_2)\, Y_n}\ \mathrm{Step1},\ \{a''[t],\ a'[t],\ a[t]\},\ \mathrm{Together}\right]$$

$$\mathrm{Final0} = \mathrm{Collect}\left[\frac{1}{2} \left(-2\, P_1 + 2\, P_2 + \left(-3\, \dot{R}^2 - 2\, R\, \ddot{R} + R^2\, \dot{F}_1 + 2\, R\, \dot{R}\, F_1\right) \rho_1 + \left(3\, \dot{R}^2 + 2\, R\, \ddot{R} - R^2\, \dot{F}_2 - 2\, R\, \dot{R}\, F_2\right) \rho_2\right),\ \{\ddot{R},\ \dot{R},\ R\}\right]$$

$$\mathrm{Final1} = \mathrm{Collect}\left[\frac{\dot{a}\left((1+n)\left(3\,\dot{R} - R\,F_1\right)\rho_1 + n\left(3\,\dot{R} - R\,F_2\right)\rho_2\right)}{(1+n)\,R\,\rho_1 + n\,R\,\rho_2} + \frac{R\,\ddot{a}}{R} + \frac{a\,(1+n)\left((2+n)\,\ddot{R} - R\,\dot{F}_1 - \dot{R}\,F_1\right)\rho_1 - a\,n\left((-1+n)\,\ddot{R} + R\,\dot{F}_2 + \dot{R}\,F_2\right)\rho_2}{(1+n)\,R\,\rho_1 + n\,R\,\rho_2},\ \{\ddot{a},\ \dot{a},\ a\}\right]$$

Out[824]=

$$\frac{1}{2}(-2\,P_1 + 2\,P_2) + \frac{1}{2} R\,\ddot{R}\,(-2\,\rho_1 + 2\,\rho_2) + \frac{1}{2}\dot{R}^2\,(-3\,\rho_1 + 3\,\rho_2) + \frac{1}{2} R^2 \left(\dot{F}_1\,\rho_1 - \dot{F}_2\,\rho_2\right) + \frac{1}{2} R\,\dot{R}\,(2\,F_1\,\rho_1 - 2\,F_2\,\rho_2)$$

Out[825]=

$$\ddot{a} + \frac{\dot{a}\left((1+n)\left(3\,\dot{R} - R\,F_1\right)\rho_1 + n\left(3\,\dot{R} - R\,F_2\right)\rho_2\right)}{(1+n)\,R\,\rho_1 + n\,R\,\rho_2} + a\left(\frac{(1+n)\left((2+n)\,\ddot{R} - R\,\dot{F}_1 - \dot{R}\,F_1\right)\rho_1}{(1+n)\,R\,\rho_1 + n\,R\,\rho_2} - \frac{n\left((-1+n)\,\ddot{R} + R\,\dot{F}_2 + \dot{R}\,F_2\right)\rho_2}{(1+n)\,R\,\rho_1 + n\,R\,\rho_2}\right)$$

D.3 P. Amendt (2003) and H. Lin (2002)

```
In[839]:= Format[D[f_[t], {t, n_ /; n < 3}]] := OverDot[f, n]
Format[R[t]] := R
Format[a[t]] := a
Format[F₂[t]] := F₂
rₛ := R[t] + a[t] Yₙ;
ϕ₁[t] := -(r^2 R'[t])/(2 R[t]) + (r^n Yₙ)/(n R[t]^(n-1)) ((a[t] R'[t])/R[t] - a'[t]);
ϕ₂[t] := (R[t]^3)/(3 r) ((3 R'[t])/R[t] + F₂[t]) + r^2/6 F₂[t] +
    (R[t]^(n+2) Yₙ)/((n + 1) r^(n+1)) (a'[t] + (2 a[t] R'[t])/R[t] + a[t] F₂[t]);
Eval1 := Simplify[Series[∂ₜ ϕ₁[t] /. r → rₛ, {Yₙ, 0, 1}]]
Eval2 := Simplify[Series[∂ₜ ϕ₂[t] /. r → rₛ, {Yₙ, 0, 1}]]
Eval3 := Simplify[Series[∂ᵣ ϕ₁[t] /. r → rₛ, {Yₙ, 0, 1}]]
Step1 := Collect[P₂ + ρ₂ (Eval2 - 1/2 (Eval3)^2) -
    (P₁ + ρ₁ (Eval1 - 1/2 (Eval3)^2)),
    {a''[t], a'[t], a[t]}, Together]
Step2 := Collect[(n (n + 1))/(R (ρ₁ + n ρ₁ + n ρ₂) Yₙ) Step1,
    {a''[t], a'[t], a[t]}, Together]
Final0 = FullSimplify[Collect[1/2 (-2 P₁ + 2 P₂ + R R̈ ρ₁ +
    (3 Ṙ^2 + 2 R R̈ + R^2 Ḟ₂ + 2 R Ṙ F₂) ρ₂), {R̈, Ṙ, R}]]
Final1 = Collect[(n ȧ (3 Ṙ + R F₂) ρ₂)/(R ((1 + n) ρ₁ + n ρ₂)) + (R ä)/R +
    (a n (R Ḟ₂ + Ṙ F₂) ρ₂ + a (-1 + n) R̈ ((1 + n) ρ₁ - n ρ₂))/((1 + n) R ρ₁ + n R ρ₂),
    {ä, ȧ, a}]
```

Out[851]=
$$\frac{1}{2}\left(-2 P_1 + 2 P_2 + R \ddot{R} \rho_1 + \left(3 \dot{R}^2 + 2 R \ddot{R} + R^2 \dot{F}_2 + 2 R \dot{R} F_2\right) \rho_2\right)$$

Out[852]=
$$\ddot{a} + \frac{n \dot{a} \left(3 \dot{R} + R F_2\right) \rho_2}{R \left((1+n) \rho_1 + n \rho_2\right)} + a \left(\frac{n \left(R \dot{F}_2 + \dot{R} F_2\right) \rho_2}{(1+n) R \rho_1 + n R \rho_2} + \frac{(-1+n) \ddot{R} \left((1+n) \rho_1 - n \rho_2\right)}{(1+n) R \rho_1 + n R \rho_2}\right)$$

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