Special Relativity in a Periodic Universe

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Abstract

In infinite flat spacetime, the specific version of the twin paradox with a stay at home twin and rocket twin, is resolved due to the inertial traveller changing direction, indicating fundamental asymmetry. By imposing boundary conditions we create a framework of compact space and counteract the need for a change of direction. Instead, resolution relies on selection of a preferred frame, determined by the topology chosen and recognising the asymmetry between observers that follows. The asymmetry between worldlines arises through the topological invariance of their homotopy classes. This illustrates how special relativity in such cases is only valid locally, hence preferred inertial frames experience more proper time than any other. In a Friedmann described universe, both the expansion of the universe, and the existence of multiply connected topology for constant time hypersurfaces, break Poincaré invariance. This singles out the same preferred inertial observer as before, who will age more quickly than her twin, comoving with the cosmic fluid.

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Part I

Background

1 Introduction

Albert Einstein, in his seminal paper on Special relativity in 1905 [1], predicted that a clock moving away in a spaceship and returning, would be found to lag behind a clock at rest. In 1911, he restated his argument:

> "*If we placed a living organism in a box... one could arrange that the organism, after any arbitrary lengthy flight, could be returned to its original spot in a scarcely altered condition, while corresponding organisms which had remained in their original positions had already long since given way to new generations. For the moving organism the lengthy time of the journey was a mere instant, provided the motion took place with approximately the speed of light.*" [6]

In the same year, Paul Langevin posited the twin paradox [2] as we know it today. The twin paradox is the most known thought experiment associated with relativity. A paradox in scientific terms refers to results which arrive in contradiction. The twin paradox is not in fact a paradox, and neither Einstein nor Langevin intended the idea to be formulated as such.

Usually the scenario plays out with a pair of twins, commonly an astronaut and earthbound pair, who come to a contradiction when measuring each others elapsed time. The astronaut leaves earth and returns back to his home twin to find that he has aged less, yet from his perspective using the same argument he sees the homebound twin move away from him, and he is stationary. Thus he concludes that she has aged less and what looked like a symmetrical idea, arrives to a paradox.

The twin paradox suggests a limitation of the principle of relativity. Independent observers only have symmetry of measurement in inertial reference frames. An inertial reference frame is a coordinate system in space and time, such that observers can identify uniform motion as opposed to accelerated motion. These frames are those which move at a constant velocity where no acceleration can be felt. Twin observers in such frames can not deduce which of the two are moving with respect to each other.

In classical mechanics we assume that the passage of time is the same in all inertial reference frames. This is Galilean relativity and the usual formulae apply here where space and time intervals are invariant, corroborating Newton's concept of absolute space and time. Special relativity on the other hand, assumes that the speed of light to any non-accelerating observer is the same in all reference frames. Mathematical transformations between inertial frames are given by the Lorentz transformations and as consequence show us that space and time are not absolute but elastic. The space and time intervals in fact depend on the relative velocity between the observer and an event he observes.

From the Lorentz formulae a clock in motion ticks slower and distances appear shorter than a clock at rest. These effects are formally called time dilation[1] and length contraction[2]. As both of these quantities are determined via the relationship between velocity and the speed of light,

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},$$

the effect only manifests itself for velocities approaching the speed of light. Such speeds are called relativistic. If a photon could carry a massless clock, no time would pass as its proper time is always zero.

In this investigation we aim to impose boundary conditions and create a framework of compact space to show that the resolution is much more of a subtle one, but provides us insight about the limitations of equivalence between inertial reference frames. Therefore, because of this new framework, it is important to understand the global properties of

[1]Moving clocks tick slower.
[2]Moving clocks appear shorter.

such spacetime.

Topology is an extension of geometry which describes space, illustrating its features, dimensions, connectivity, orientability, infiniteness or finiteness, without measurement. Continuity is important in such a space, and can be visioned such that space can stretch without tearing and ends be glued together. Space can be stretched, crushed or kneaded in any way without changing its topology. We are primarily concerned with space of zero curvature so that special relativity still applies. Our spacetime occupies the infinite Euclidean plane and four others can be constructed from this which will later be shown.

For completeness I have shown a basic example of the Twin Paradox in *The Classical Twin Paradox* (Section 3.2). This leads us to an intuitive result when the rules of special relativity are understood. For a journey to be described by special relativity, inertial reference frames must be maintained and a change of direction violates this principle. The asymmetry discovered here is a more obvious one than the compact space we aim to consider. Ultimately the reader will appreciate the subtleties of a compact spacetime resolution, suggesting the idea of a preferred reference frame in which calculation of time is trivial. This provides us with insight on what observers can expect to understand about the compact universe they reside in.

Basic cylinder construction of our new spacetime is discussed in *Defining the Periodic Universe* (Section 3.3). Later mappings are shown in *Constructing the* 3*D Torus* (Section 4.3) and *Constructing the* 4*D Hypertorus* (Section 4.4).

The punchline ideas are explored in *Periodic Twin Paradox* (Section 3.5), *Shining a Torch in the Dark* (Section 3.6) and *Both Twins as Inertial Observers* (Section 3.7). Preferred reference frames and formulas for time difference depending of the period L and velocity v are discovered providing observers the ability to determine such quantities given valid knowledge of either variable.

2 Overview of Special Relativty

2.1 The Principles Of Special Relativity

Special Relativity can be deduced from two postulates:

- The Principle of Relativity (Galileo): No experiment can measure the absolute velocity of an observer; the results of any experiment performed by an observer do not depend on his speed relative to other observers who are not involved in the experiment.

- The Constancy of the speed of light (Einstein): The speed of light relative to any non-accelerating observer is $c = 3 \times 10^8 ms^{-1}$, regardless of the motion of the light source relative to the observer.

These postulates infer that there is no sense in defining absolute motion, or rest, as uniform motion is undetectable. The finite limit for the speed of light is itself what causes relativistic effects to appear for the material universe. [5, 8]

2.2 Definition of an Inertial Observer in Special Relativity

An observer can be defined as a recorder of both space and time of a given event formally written as $\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$, however in general we will be dealing with flat space-time $\binom{t}{x}$ initially. For such a coordinate system to be inertial it must satisfy the following:

- The distance between a point $P_1(x_1)$ and $P_2(x_2)$ is independent of time.

- Clocks that sit at every point passing time are synchronized and run at the same rate.

- The geometry of space at any constant time is Euclidean.

Note however that acceleration is not mentioned. The consequences of which will be explored later, and show that only non-accelerating observers can keep their clocks synchronized.

An *observation* made by an inertial observer is defined as recording $\binom{t}{x}$ of an event in the same space and time. In other words the clock reading must be taken precisely in the location of the event to be called an inertial observer. This observation made by an inertial observer is what we will also refer to as *proper time*. [5, 8]

2.3 Units for Special Relativity

Because the speed of light is now clarified as fundamental it makes sense to redefine the way that we measure space and time. If we define $c = 1$ the new unit of time is the meter. One meter of time is then the time taken for light to travel one meter. Illustrating this we have:

$$c = \frac{\text{distance light travels in any given time interval}}{\text{time taken for light to travel one meter}} = \frac{1\text{m}}{1\text{m}} = 1$$

It follows from $c = 3 \times 10^8 ms^{-1} = 1$ that

$$1\text{s} = 3 \times 10^8\text{m and } 1\text{m} = \frac{1}{3 \times 10^8\text{s}}$$

From this we find that Energy and Momentum are now measured in kg, Acceleration in m^{-1} and Force in kgm^{-1}. [5, 8]

2.4 Spacetime Diagrams and Notational Conventions

A single point fixed by x and t is called an *event*. A line in space with relation $x = x(t)$ is known as a particles *world line*, the gradient of which is $\frac{\text{d}t}{\text{d}x} = \frac{1}{v}$. A light ray always travels at a 45° line on a space-time diagram.

For all the work to follow we must be clear and concise with description, thus I will define such notation here for reference.

- Events will use cursive capitals e.g. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{O}$ will be used to describe observers. Later, in context, I will be using observers Jack and Jill who

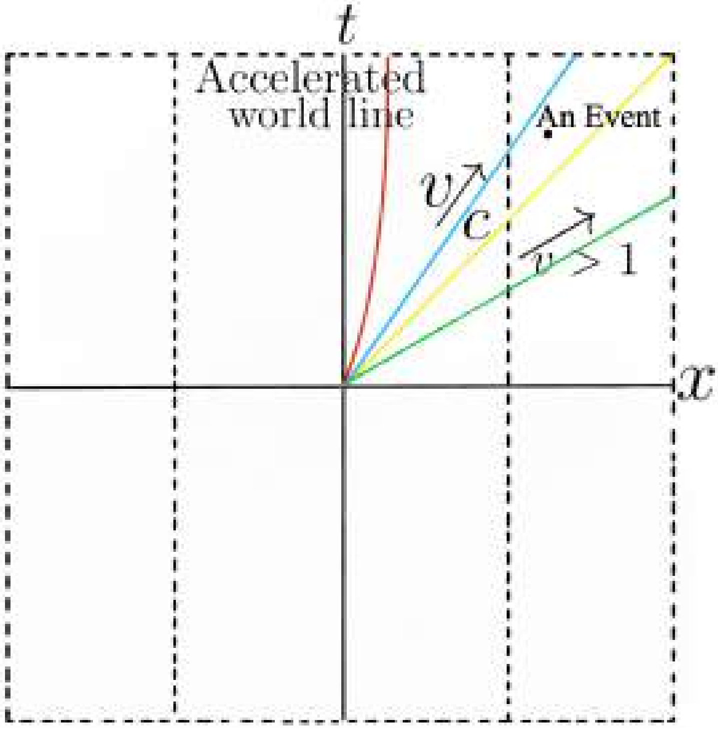


Figure 1: A space-time diagram in natural units. [8]

will be defined as $\tilde{\mathcal{O}}_{\text{Jack}}$ and $\mathcal{O}_{\text{Jill}}$ respectively.

- The coordinates $\binom{t}{x}$ are all measured in meters.

- The Greek index referred to as x^{α} can take a value from the set $0, 1$ thus $\binom{t}{x}$ would correspond to $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$ in this case. If I state x^{α} by itself I am referring to all the coordinates implied.

- The Latin index referred to as x^{i} distinguishes spacial coordinates only where (x) would correspond to (x^1).

[8]

2.5 Constructing Multiple Observer Spacetime Diagrams

To illustrate both observers in a single diagram, we begin with $\mathcal{O} \mapsto \binom{t}{x}$ and $\tilde{\mathcal{O}} \mapsto \begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix}$, who travels with velocity v in the x direction relative to $\mathcal{O}$.

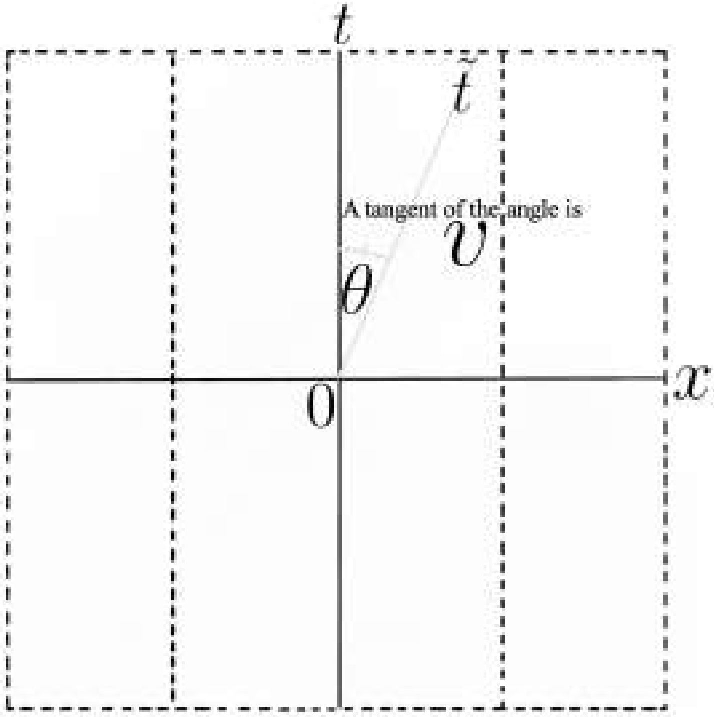
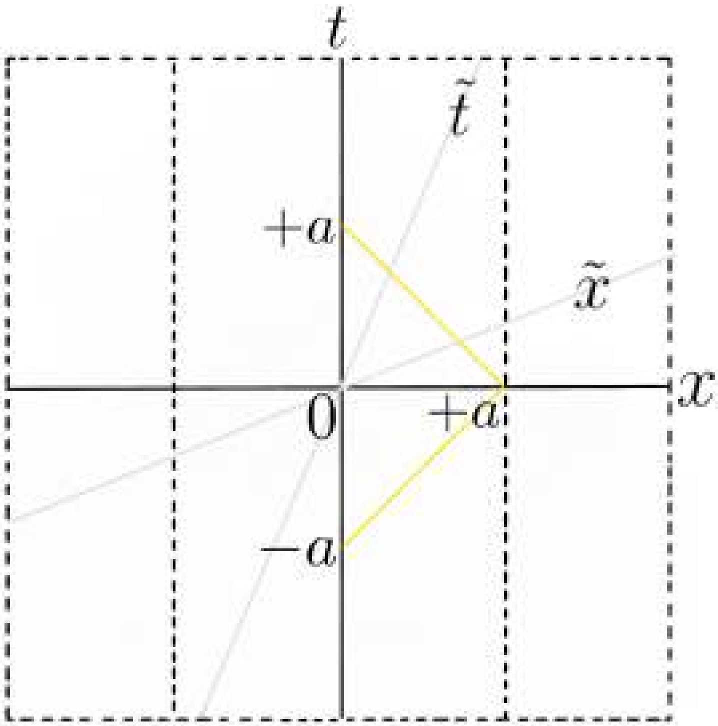
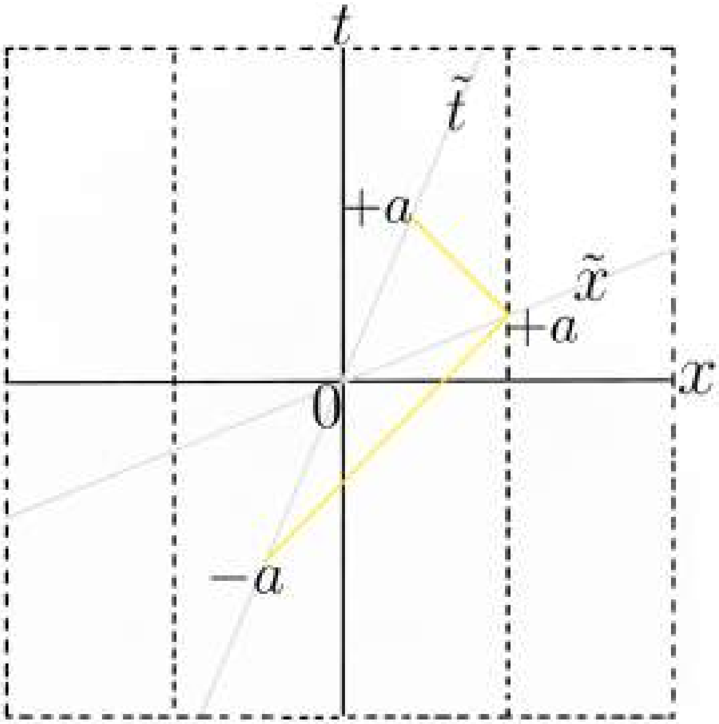


Figure 2: The $\tilde{t}$ axis is the locus of events at constant $\tilde{x} = 0$ which is the locus of the origin of $\tilde{\mathcal{O}}$'s spatial coordinates. This is $\tilde{\mathcal{O}}$'s world line.

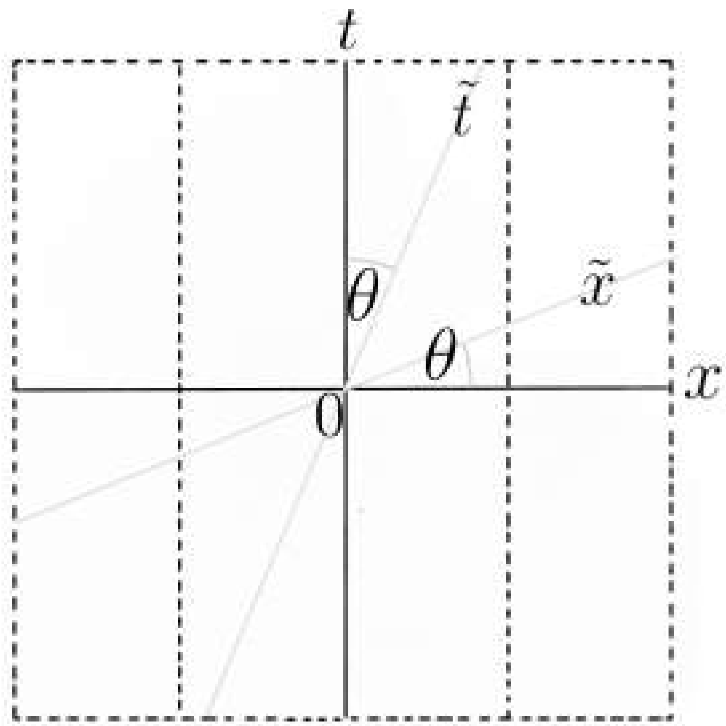


(a) Light reflected at a, as measured by $\tilde{\mathcal{O}}$.

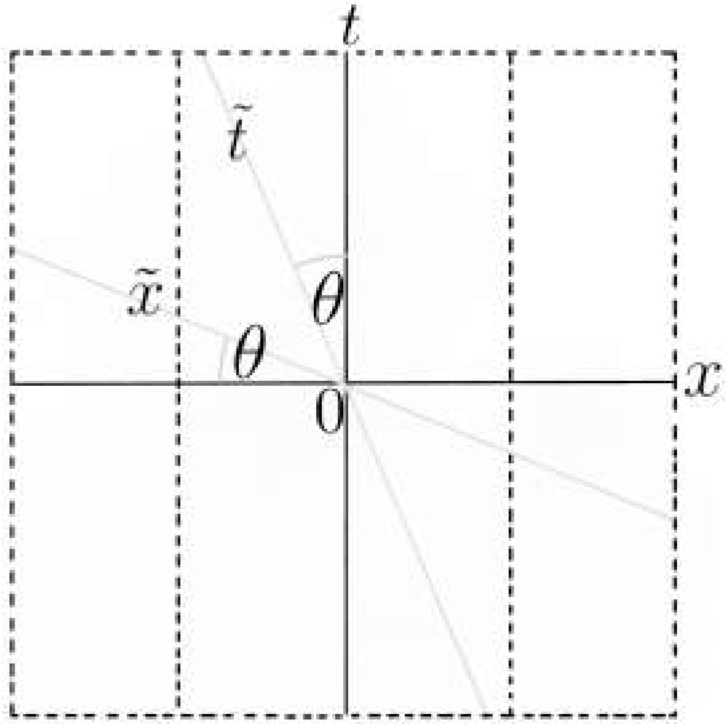


(b) The reflection as measured by $\mathcal{O}$.

Figure 3: The $\tilde{x}$ axis is the locus of events that reflect light rays such that they return to the $\tilde{t}$ axis at $+a$ if they left it at $-a$ for all a. Geometrically this demonstrates the second postulate of Special Relativity. Events simultaneous to $\tilde{\mathcal{O}}$ are not simultaneous to $\mathcal{O}$. In other words the lines from respective frames $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are not parallel.



(a) $\mathcal{O}$



(b) $\tilde{\mathcal{O}}$

Figure 4: The space-time diagrams represent the same physical situation. The one on the left is the space-time diagram $\mathcal{O}$, in which $\tilde{\mathcal{O}}$ moves to the right. The one on the right is drawn from the point of view of $\tilde{\mathcal{O}}$, in which $\mathcal{O}$ moves to the left. All four angles are equal to $\tan^{-1}|v|$ where $|v|$ is the relative speed of $\mathcal{O}$ and $\tilde{\mathcal{O}}$.

2.6 Interval Invariance

The final construction of $\mathcal{O}$'s coordinates require us to define length scale. The interval between any two events is separated by coordinates $(\Delta t, \Delta x)$ such that

$$\Delta^2 s = -(\Delta t)^2 + (\Delta x)^2$$

and on assumption the relation between $\mathcal{O}$ and $\tilde{\mathcal{O}}$ is linear we also have

$$\Delta^2 \tilde{s} = -(\Delta \tilde{t})^2 + (\Delta \tilde{x})^2$$

$\Delta \mathrm{s}^2$ is a property only of two events and not of the observer. It is used to classify the relation between events.

$$\Delta s^2 \begin{cases} > 0, \text{ Spacelike (spatial increments dominate)} \\ < 0, \text{ Timelike (time increments dominate)} \\ = 0, \text{ Lightlike (events lie on the same path)} \end{cases} \tag{2.1}$$

For example, all world lines of physical objects move in a timelike direction. The constancy of the speed of light implies that the intervals Δs^2 and $\Delta \tilde{s}^2$ between any two events as computed by different observers satisfy the relation

$$\Delta \tilde{s}^2 = \Delta s^2 \tag{2.2}$$

which means the the interval is independent of the observer. [8]

2.7 Hyperbolae Invariance

Next we can calibrate the $\tilde{\mathcal{O}}$ axes in the $\mathcal{O}$ space-time diagram with the equation

$$-t^2 + x^2 = a^2,\ a \in \mathbb{R}.$$

The hyperbola in the space-time diagram of $\mathcal{O}$ passes through events whose interval from the origin is a^2. By the invariance of the interval, the same events have interval a^2 from the origin in $\tilde{\mathcal{O}}$ which means that they lie on the curve

$$-\tilde{t}^2 + \tilde{x}^2 = a^2,\ a \in \mathbb{R},$$

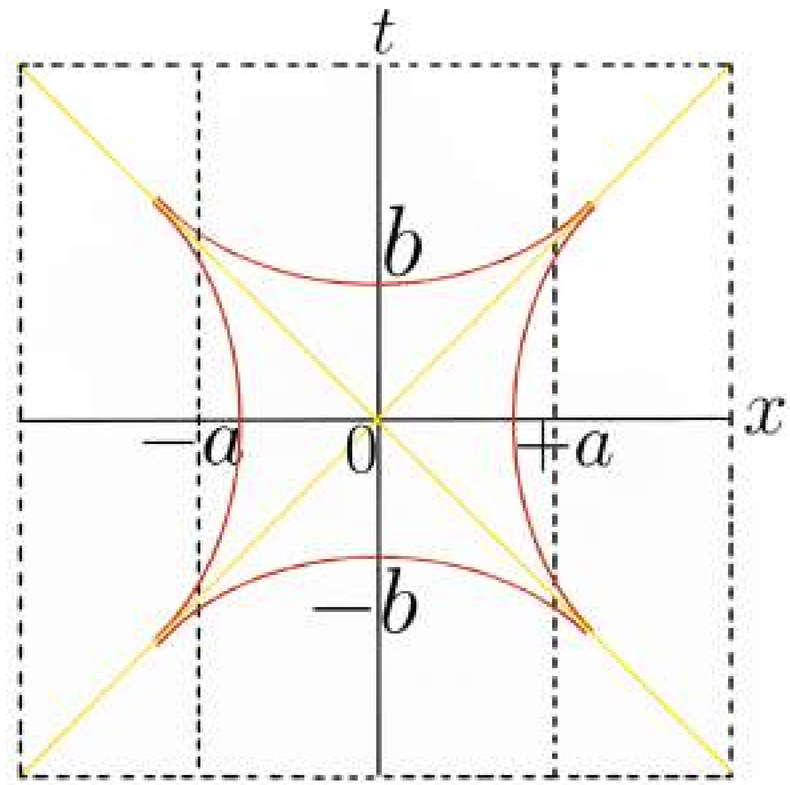
which is space-like separated from the origin. In a similar fashion, events on the curve

$$-t^2 + x^2 = -b^2,\ b \in \mathbb{R}$$

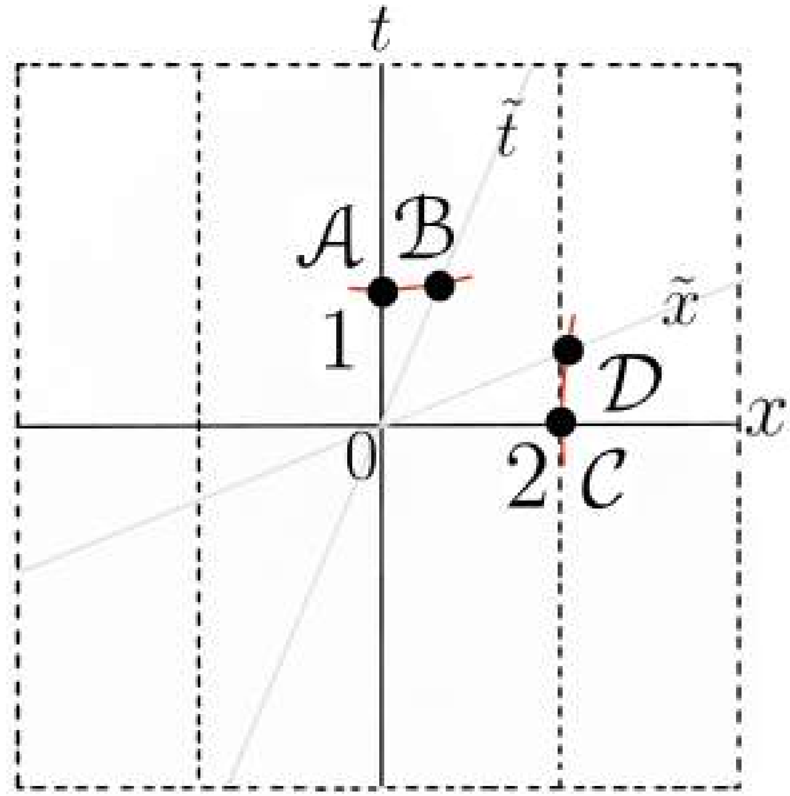
all have a time-like interval $-b^2$ from the origin, and also lie on the curve

$$-\tilde{t}^2 + \tilde{x}^2 = -b^2,\ b \in \mathbb{R}.$$

The tangent to a hyperbola at any event is a line of simultaneity of the Lorentz frame whose time axis joins the event to the origin. [8]



(a) Invariant hyperbolae, for $a > b$.



(b) Using the hyperbolae through events $\mathcal{A}$ and $\mathcal{C}$ to calibrate the $\tilde{x}$ and $\tilde{t}$ axes.

Figure 5

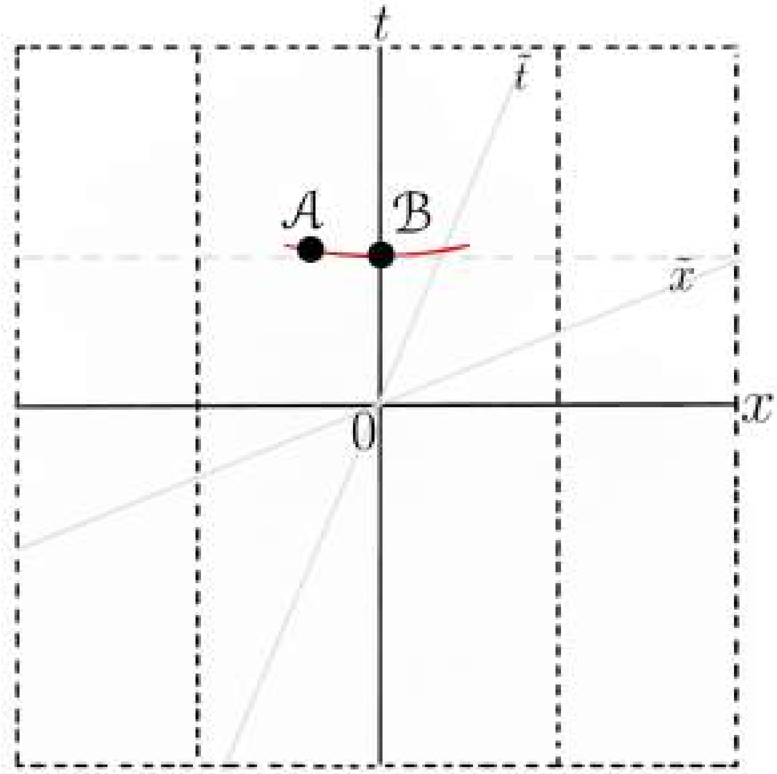
2.8 Time Dilation

From Figure 5b, it follows that when a clock moves on the $\tilde{t}$ axis and reaches event $\mathcal{B}$, it has a reading of $\tilde{t} = 1$, with coordinate $t = \frac{1}{\sqrt{1-v^2}}$ in $\mathcal{O}$. From the perspective of $\mathcal{O}$ the clock appears to run slow by

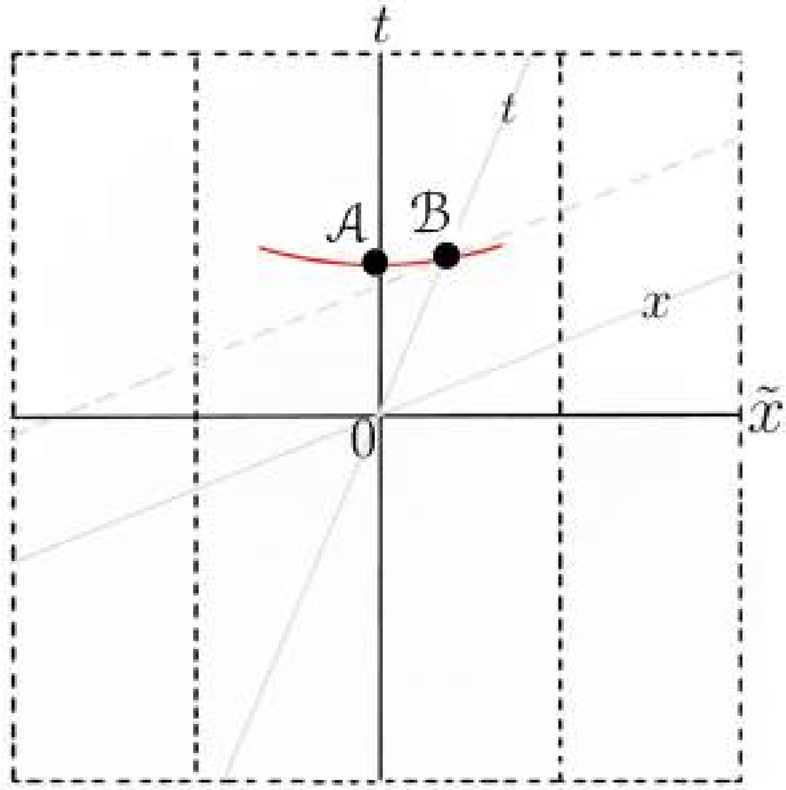
$$(\Delta t)_{\mathcal{O}} = \frac{(\Delta \tilde{t})_{\tilde{\mathcal{O}}}}{\sqrt{1 - v^2}}, \tag{2.3}$$

where $\Delta\tilde{t}$ is the time measured by a single clock which moves on a world line from the origin to event $\mathcal{B}$. Δt is the difference in readings of two clocks at rest in $\mathcal{O}$, where one clock is on a world-line through the origin, and one is on a world-line passing through event $\mathcal{B}$. The *proper time* between events $\mathcal{B}$ and the origin is defined as the time ticked off by a clock which passes through both events. The *proper time* is just the square root of the negative of the interval. [8]

$$\Delta s^2 = -\Delta\tilde{t}^2 = -\Delta\tau^2 \tag{2.4}$$



(a) $\mathcal{O}$



(b) $\tilde{\mathcal{O}}$

Figure 6: **(a)** A line of simultaneity in $\mathcal{O}$ is a tangent to the hyperbola at $\mathcal{A}$. **b** The same tangent as seen by $\tilde{\mathcal{O}}$.

2.9 Lorentz Contraction

Figure 7 shows the world path of a rod at rest in $\tilde{\mathcal{O}}$. Its length in $\tilde{\mathcal{O}}$ is the square root of $\Delta s^2_{\mathcal{AC}}$, while its length in $\mathcal{O}$ is the square root of $\Delta s^2_{\mathcal{AB}}$. If event $\mathcal{C}$ has coordinates $\tilde{t} = 0$, $\tilde{x} = l$ then it has x coordinate in $\mathcal{O}$

$$x_{\mathcal{C}} = \frac{l}{\sqrt{1-v^2}}.$$

Since the $\tilde{x}$ axis is represented by the line $t = vx$,

$$t_{\mathcal{C}} = \frac{vl}{\sqrt{1-v^2}}.$$

The line $\mathcal{BC}$ has the gradient $\frac{\Delta x}{\Delta t} = v \Rightarrow \frac{x_{\mathcal{C}} - x_{\mathcal{B}}}{t_{\mathcal{C}} - t_{\mathcal{B}}}$ and we want to find $x_{\mathcal{B}}$ when $t_{\mathcal{B}} = 0$. Thus

$$x_{\mathcal{B}} = x_{\mathcal{C}} - vt_{\mathcal{C}} = \frac{l}{\sqrt{1-v^2}} - \frac{v^2 l}{\sqrt{1-v^2}} = l\sqrt{1-v^2} \tag{2.5}$$

which we will hence define as the *Lorentz Contraction.* [8]

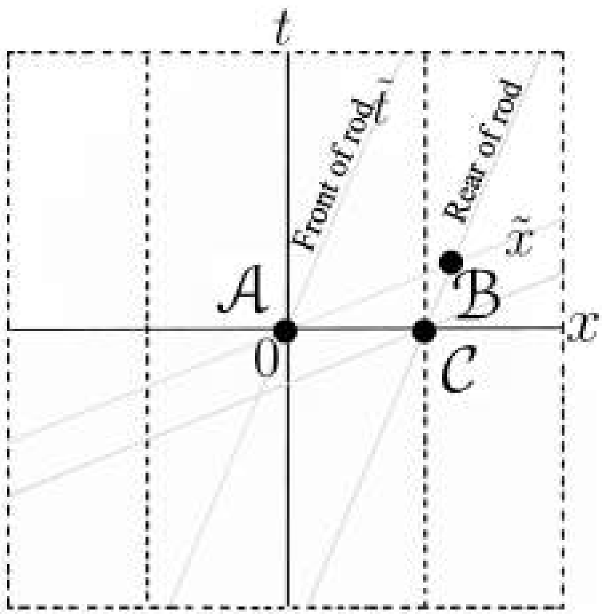


Figure 7: The proper length of $\mathcal{AC}$ is the length of the rod in its rest frame, while that of $\mathcal{AB}$ is its length in $\mathcal{O}$.

2.10 The Lorentz Transformation

The *Lorentz Transformation* expresses the coordinates of $\tilde{\mathcal{O}}$ in terms of those of $\mathcal{O}$. The axes are oriented such that $\tilde{\mathcal{O}}$ moves with speed v on the positive x axis relative to $\mathcal{O}$. The lengths perpendicular to the x axis are the same regardless of measurement by $\mathcal{O}$ or $\tilde{\mathcal{O}}$. The general linear transformation is then

$$\tilde{t} = \alpha t + \beta x \qquad \tilde{y} = y$$
$$\tilde{x} = \gamma t + \sigma x \qquad \tilde{z} = z$$

where α, β, γ, σ depend only on v. From Fig. (1.4) the $\tilde{t}$ and $\tilde{x}$ axes have equations

$$\tilde{t}_{\tilde{x}=0} : vt - x = 0,$$
$$\tilde{x}_{\tilde{t}=0} : vx - t = 0.$$

which implies

$$\frac{\gamma}{\sigma} = -v, \qquad \frac{\beta}{\alpha} = -v$$

of which follows the transformation

$$\tilde{t} = \alpha(t - vx),$$
$$\tilde{x} = \sigma(x - vt).$$

From Figure 3 we can also see that the two events $(\tilde{t} = 0, \tilde{x} = a)$ and $(\tilde{t} = a, \tilde{x} = 0)$ are connected by a light ray which means that $\alpha = \sigma$. From this we have

$$\tilde{t} = \alpha(t - vx),$$
$$\tilde{x} = \alpha(x - vt).$$

The *invariance of the interval* states that

$$-(\Delta\tilde{t})^2 + (\Delta\tilde{x})^2 = -(\Delta t)^2 + (\Delta x)^2 \qquad (2.6)$$

which gives us

$$\alpha = \pm\frac{1}{\sqrt{1-v^2}}$$

of which we select the + sign so that when $v = 0$ we arrive at an identity rather than inverting the coordinates.

The *Lorentz Transformations* are then

$$\begin{aligned}
\tilde{t} = \frac{t}{\sqrt{1-v^2}} - \frac{vx}{\sqrt{1-v^2}} &= \gamma(t - vx) = t\cosh\theta - x\sinh\theta, \\
\tilde{x} = -\frac{vt}{\sqrt{1-v^2}} + \frac{x}{\sqrt{1-v^2}} &= \gamma(x - vt) = -t\sinh\theta + x\cosh\theta, \\
t &= \gamma(\tilde{t} + v\tilde{x}) = \tilde{t}\cosh\theta + \tilde{x}\sinh\theta, \\
x &= \gamma(\tilde{x} + v\tilde{t}) = \tilde{t}\sinh\theta + \tilde{x}\cosh\theta, \\
\tilde{u} &= \frac{u-v}{1-uv}, \\
u &= \frac{\tilde{u}+v}{1+\tilde{u}v}, \\
v &= \tanh\theta, \\
\tilde{y} &= y, \\
\tilde{z} &= z.
\end{aligned} \tag{2.7}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\theta$ and θ is shown in Figure 8.

which give the simplest relation between the coordinates of $\tilde{\mathcal{O}}$ and $\mathcal{O}$. To use this however, requires the spatial coordinates to be oriented in a specific way: $\tilde{\mathcal{O}}$ must move with speed v in the positive x direction as seen by $\mathcal{O}$, and the axes of $\tilde{\mathcal{O}}$ must be parallel to the corresponding ones in $\mathcal{O}$. [8]

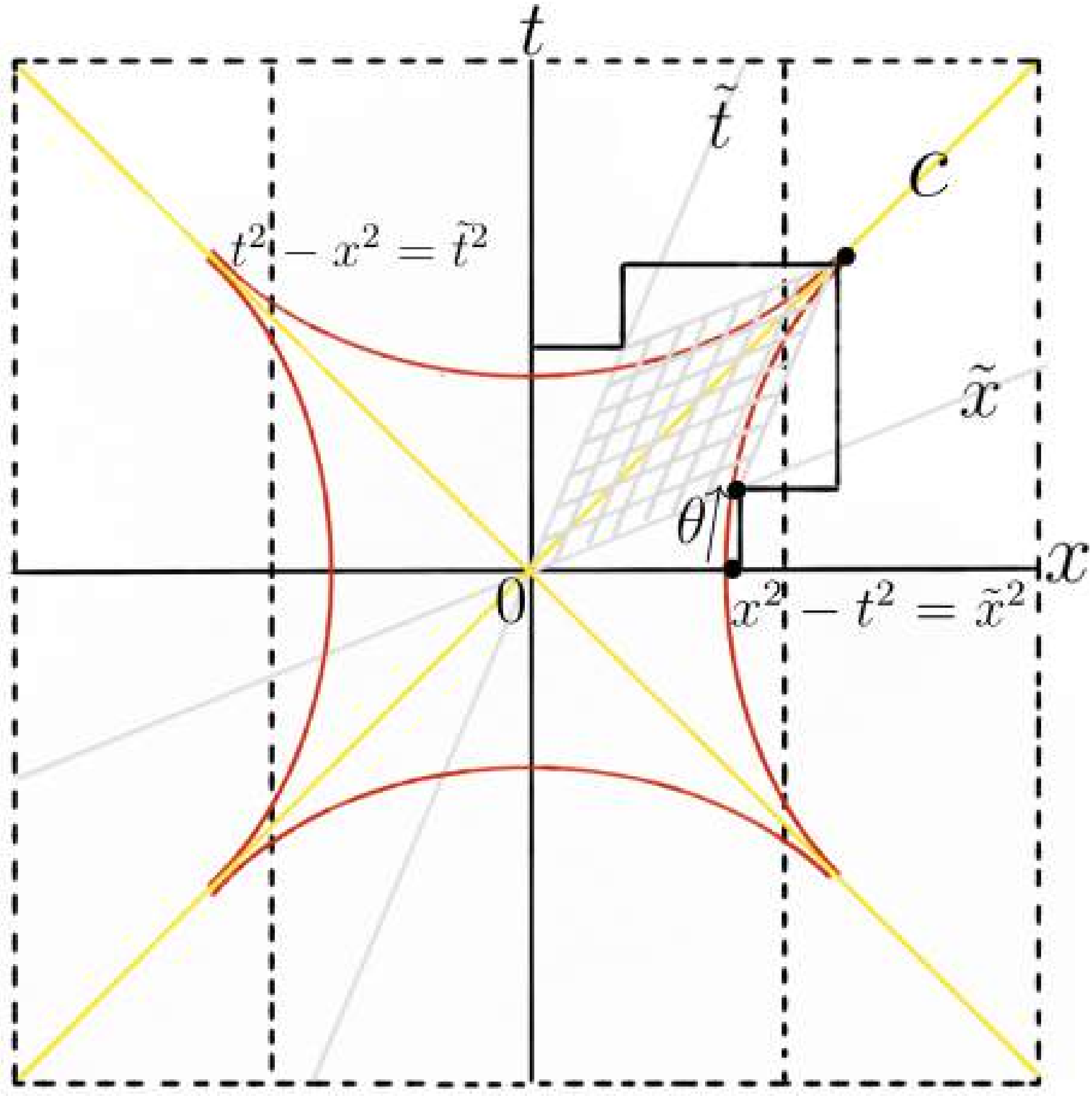


Figure 8: $\mathcal{O}$'s rest frame. An illustration of all the steps combined so far. The lines of simultaneity are shown for the alternate observer in both diagrams.

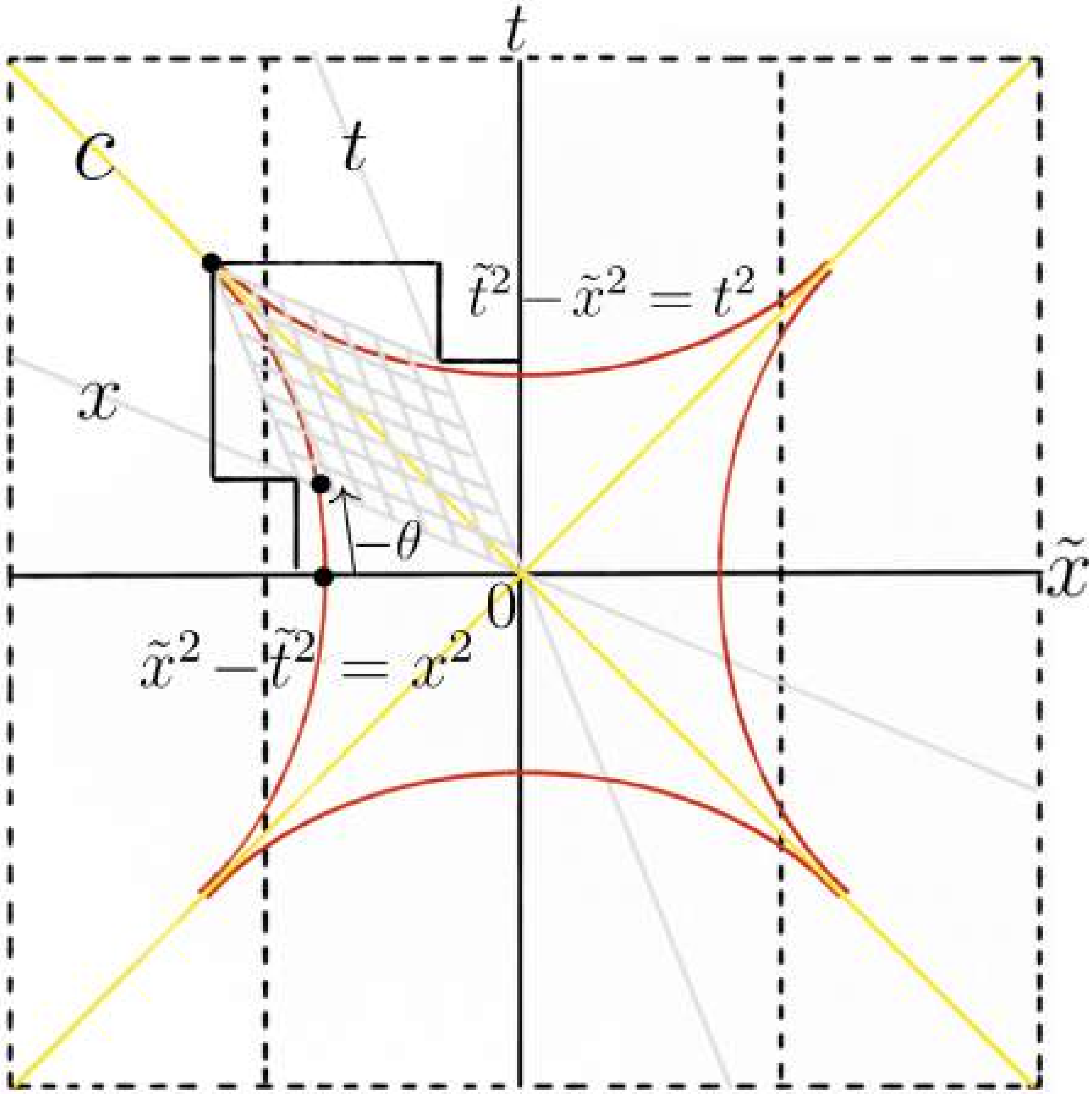


Figure 9: $\mathcal{O}$'s rest frame. An illustration of all the steps combined so far. The lines of simultaneity are shown for the alternate observer in both diagrams.

Part II

Special Relativity in a Periodic Universe

3 Periodic Spacetime in $2D$

3.1 Proper Time

The most important concept that needs to be understood is the idea of *proper time*. Langevin in 1911 introduced the twin paradox as we know it today to discuss the implications of special relativity. The differential ageing between the twins can be demonstrated by calculating the proper time τ along two separate paths shown in Figure 10. For constant speeds the proper time of an observer is calculated by integrating

$$d\tau = \frac{dt}{\gamma}. \tag{3.1}$$

For an observer at rest we have

$$d\tau = dt. \tag{3.2}$$

As $\gamma \leq 1$, the proper time measured along an inertial frame will be less than the measured time in a rest frame, where it should be noted that rest frames are also inertial. This inequality comes from the fact that proper time is a path dependant quantity. [27]
In subsection 3.2 we end up with

$$\tau_{\mathcal{O}_{\text{Jill}}} = 10 > \tau_{\tilde{\mathcal{O}}_{\text{Jack}}} = \frac{10}{\gamma}$$

which demonstrates that the $\tilde{\mathcal{O}}_{\text{Jack}}$'s inertial frame experiences less time than $\mathcal{O}_{\text{Jill}}$'s rest frame. Later we will use a more general proper time formula, concerning the change in coordinates.

3.2 The Classical Twin Paradox

The classic version of the Twin Paradox, in a universe with no imposed boundary conditions, can easily be resolved in infinite flat space. It is important to understand the nature of this fundamental paradox, as we move on to explore the effect of closed universes on such problems. This problem is not so much of a paradox, but a misunderstanding of the fundamental principles of relativity and prior assumption.

Consider then, $\mathcal{O}_{\text{Jill}}$ at rest and $\tilde{\mathcal{O}}_{\text{Jack}}$ travelling with velocity $\tilde{v}$. $\tilde{\mathcal{O}}_{\text{Jack}}$ travels out to a distant star, at which point he turns around, returning back to $\mathcal{O}_{\text{Jill}}$. From her perspective, she determines his elapsed time to be less than her own, and thus concludes he returns home younger than her. Yet from $\tilde{\mathcal{O}}_{\text{Jack}}$'s perspective he is the one at rest, and she is moving away from him with velocity v! Thus from the same argument he also concludes upon her 'return' she is younger than he is!

The solution to this is best shown via diagrams to illustrate the rotation of time for an inertial traveller. Because of the change in direction at the turnaround point, $\tilde{\mathcal{O}}_{\text{Jack}}$ has changed velocity, so how he determines simultaneous time has in fact rotated with him. In reality you would have to undergo an acceleration to change direction like this, and this change in acceleration accounts for the apparent loss of time in the transition. When firing your rockets to change direction his notion of time very quickly rotates through the missing gap shown in Figure 10, which allows both observers to properly account for the missing time.

It is important to note, that this fundamental solution, is the one to which we will attempt to override when imposing boundary conditions.

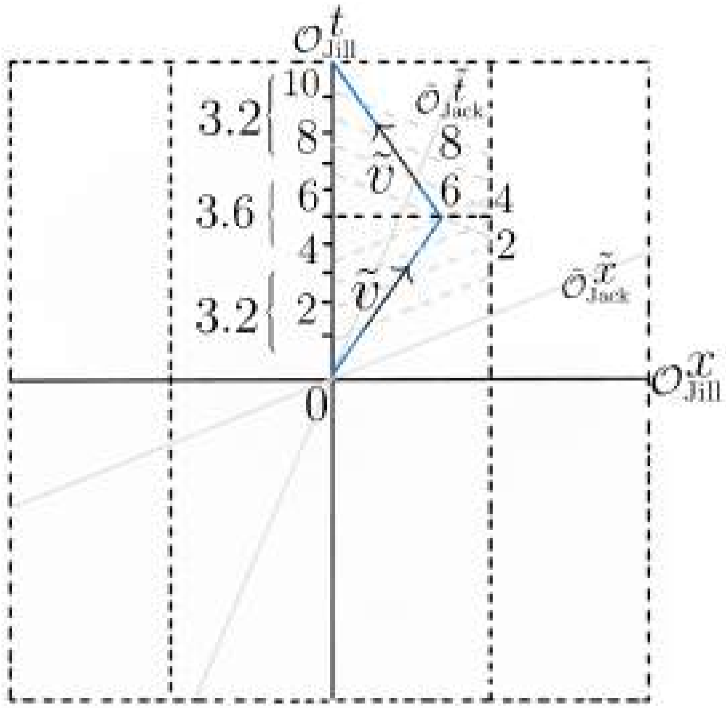


Figure 10: On both outward and inbound journeys $\mathcal{O}_{\mathrm{Jill}}$ sees $\tilde{\mathcal{O}}_{\mathrm{Jack}}$'s time pass more slowly, illustrated here measured as 3.2m. Upon the rotation at 5m he effectively skips over the 3.6m he lost in the described paradox and thus they both concur from each others perspectives that $\tilde{\mathcal{O}}_{\mathrm{Jack}}$ is in fact the younger twin upon his return.

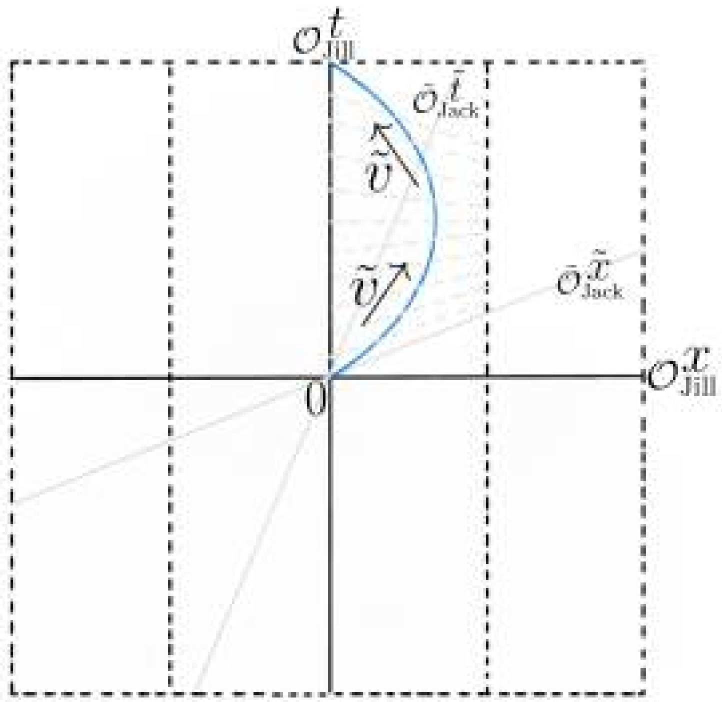


Figure 11: A more realistic, non-instantaneous, change of direction illustrates the rotation of time with the traveller.

3.3 Defining the Periodic Universe

Now we have established the classic problem, wherein one twin undergoes acceleration and the other does not. We can consider alternative additions to this apparent *paradox*, where we alter this system such that we can explore solutions to other ideas. Because our solution to the previous problem was mainly that one twin undergoes a change of acceleration, we can posit what happens when both undergo no change of acceleration and devise a $2D$ topology in which this can be achieved.

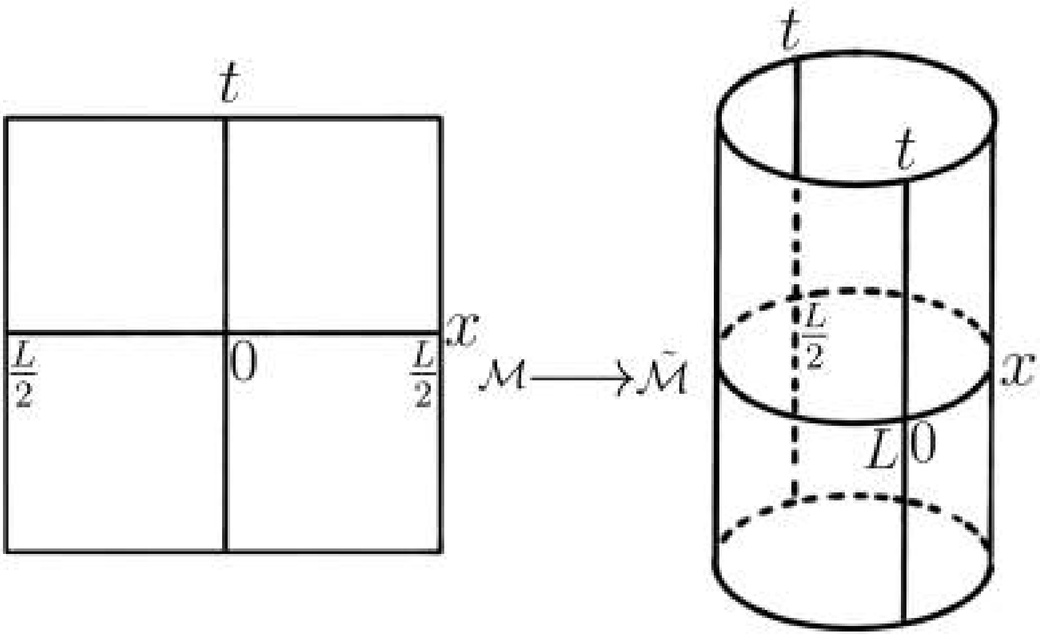


Figure 12: The mapping from a sheet of paper $\mathcal{M}$ to a cylinder $\tilde{\mathcal{M}}$ defined below.

If we define a two-dimensional universe in the shape of a cylinder, such that the distance to traverse one period is L. and time extends infinitely up the cylinder where space wraps around. Transforming our standard Minkowski space $\mathcal{M}$ from a sheet of paper to a cylinder of Minkowski space $\tilde{\mathcal{M}}$ (Figure 12) changes the topology. As this is a global change, the local structure of spacetime remains the same. We can also state that this cylinder is flat because it can be constructed from a single piece of paper, maintaining its dimensions. As only the global topology changes in this construction, we can still state that the geometry in the piece of paper is Lorentzian. Therefore Special

[3]Special Relativity only applies to space-time with zero curvature.

Relativity[3] still describes the physics governing this space-time.

The cylinder itself is formed via the rectangular region in the plane, given by the primary range

$$x \in [0, L] \text{ and } \tilde{x} \in \left[-v\tilde{t}, -v\tilde{t} + \frac{L}{\gamma}\right], \tag{3.3}$$

due to both reference frames requiring the same restrictions to their respective domains. Everything outside of this period is a repetition of each cycle, but as we will later demonstrate, asymmetries can occur in moving frames and lead us to conclude that a preferred rest frame exists in a periodic universe. [24, 26, 37]

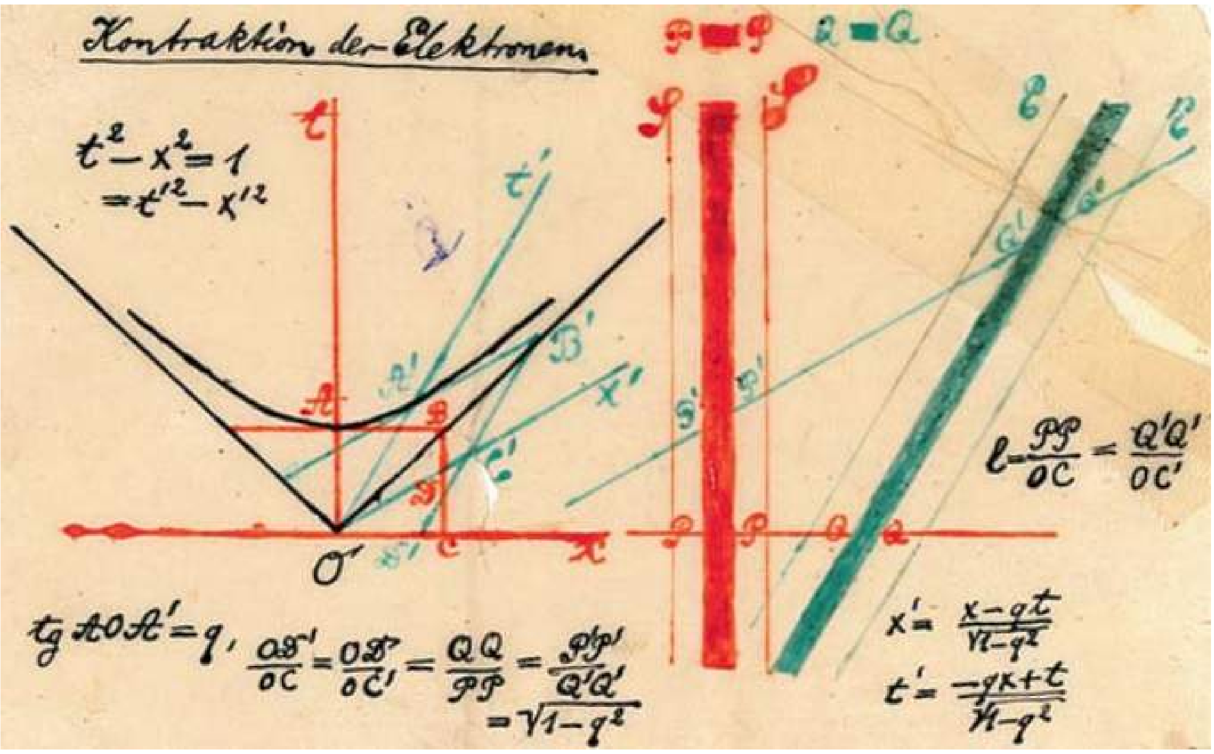


Figure 13: Hand-coloured transparency presented by Minkowski in his 1908 Raum und Zeit lecture.

3.4 The Idea of a Preferred Reference Frame

This section aims to generalize the unique consequences to Special Relativity when periodic boundary conditions are imposed on spacetime, and will hint at resolutions to later paradoxes due to the asymmetry of moving frames.

3.4.1 The Importance of Simultaneity

In Galilean relativity, identifying points simultaneously gives no preference for any frame, and the idea of simultaneous time is intuitive. With Special Relativity if we identify[4] points at the same time in $\mathcal{O}_{\text{Jill}}$'s rest frame, then in $\tilde{\mathcal{O}}_{\text{Jack}}$'s moving frame points are not identified at the same time due to the simultaneity laws of our cylindrical universe $\tilde{\mathcal{M}}$.

Theorem 1. *In a periodic universe under the spacetime of Special Relativity imposes a preferred reference frame, one which all events can be identified simultaneously.*

Theorem 2. *The events* $\mathcal{A}\binom{0}{0} = \mathcal{B}\binom{0}{L}$ *are identified simultaneously in* $\mathcal{O}_{\text{Jill}}$*'s frame.* [26]

3.4.2 Size of the Periodic Universe

A potential paradox arises when considering what happens to the universe from each perspective due to length contraction. The size of our periodic universe is the length of one cycle. This is L in $\mathcal{O}_{\text{Jill}}$'s rest frame and γL in $\tilde{\mathcal{O}}_{\text{Jack}}$'s moving frame. If $\tilde{\mathcal{O}}_{\text{Jack}}$[5] decides to run from $\tilde{x} = 0$ to $\tilde{x} = \gamma L$, laying out meter sticks end to end, he will find himself back where he started. Consequently, the right end of the last meter stick will be in contact with the left side first, due to the points being identified. As the sticks are all at rest, the difference in identified times is not relevant. This suggests our preferred frame is that in which the size of the universe is the smallest, since in $\tilde{\mathcal{O}}_{\text{Jack}}$'s frame the size is γL, and in $\mathcal{O}_{\text{Jill}}$'s frame the size is L.

If we consider the opposite scenario from $\mathcal{O}_{\text{Jill}}$'s frame, $\tilde{\mathcal{O}}_{\text{Jack}}$ would see each of her meter

[4]Identify means that they both represent the same point.

[5]He sees himself as the one at rest from his perspective even though he is our moving observer thus the time difference is not relevant when considering his perspective.

sticks length contracted by a factor of $\frac{1}{\gamma}$, suggesting he sees her universe to have a size of $\frac{L}{\gamma}$. As $\frac{L}{\gamma} < L$, $\tilde{\mathcal{O}}_{\text{Jack}}$ concludes that a larger universe is the preferred frame.

The event $\mathcal{B}$ is not at the origin of $\tilde{\mathcal{O}}_{\text{Jack}}$'s frame thus this apparent paradox is resolved in favour of the original conclusion from this asymmetry (Figure 14). In summary then we have that:

Theorem 3. *One cycle of $\mathcal{O}_{\text{Jill}}$'s frame at rest is smaller when observed from $\tilde{\mathcal{O}}_{\text{Jack}}$'s moving frame. Specifically γ^2 is the number of cycles in $\mathcal{O}_{\text{Jill}}$'s rest frame for each cycle of $\tilde{\mathcal{O}}_{\text{Jack}}$'s.*

This has application in section 3.6 where we demonstrate firing lightrays around the cylinder to the left and right from the origin as an inertial traveller. [26]

3.4.3 Clock Synchronization

When synchronizing clocks in $\tilde{\mathcal{O}}_{\text{Jack}}$'s frame, the distance between them is a factor of γL apart. Using the results from theorem 2, beginning in $\tilde{\mathcal{O}}_{\text{Jack}}$'s frame he decides to synchronise clocks to the right. After one full cycle a clock to the left of the origin will read $\tilde{t} = 0$ and be identified simultaneously to the right of the origin as $\tilde{t} = \gamma v L$, suggesting discontinuity. This means that we again have a preferred reference frame, namely $\mathcal{O}_{\text{Jill}}$'s. This is shown in figure 14.

Theorem 4. *A clock to the right of an observer will always be ahead by a factor of $\tilde{t} = \gamma v L$ from $\tilde{\mathcal{O}}_{\text{Jack}}$'s inertial frame. $\mathcal{O}_{\text{Jill}}$'s rest frame is the only frame which clocks can be synchronized globally.*

3.5 Periodic Twin Paradox

Now we have generalised issues that might arise from restricting our domain periodically, its time to come back to our original paradox and determine the consequences.

$\mathcal{O}_{\text{Jill}}$ is at rest at the event $\mathcal{A}\binom{0}{0}$ in the cylinder, and $\tilde{\mathcal{O}}_{\text{Jack}}$ leaves from event $\mathcal{A}\binom{0}{0}$ moving to the right with velocity v, accelerating initially, but later switching his engines off to undergo no more acceleration. He travels along a helix circling the cylinder, and makes a full pass of the universe at $t = \frac{L}{v}$ recorded bu $\mathcal{O}_{\text{Jill}}$.

When $\tilde{\mathcal{O}}_{\text{Jack}}$ and $\mathcal{O}_{\text{Jill}}$ compare clocks, $\mathcal{O}_{\text{Jill}}$ finds that the elapsed time[6] for $\tilde{\mathcal{O}}_{\text{Jack}}$'s journey is $t = \frac{L}{v}$, where from $\tilde{\mathcal{O}}_{\text{Jack}}$'s perspective the elapsed time is less, and he measures the journey as $\tilde{t} = \frac{L}{\gamma v}$. The issue here is that both observers have travelled with constant uniform velocity but are in disagreement on elapsed time.

The resolution then, is best solved like all relativistic paradoxes by considering the space-time diagrams of both individual observers. Lorentz transformations act on the whole of $\mathcal{M}$ usually, but when we apply the same reasoning to $\tilde{\mathcal{M}}$ the transformations only make sense locally. Without frames of reference where there are surfaces of simultaneity this approach fails. In $\tilde{\mathcal{M}}$ only $\mathcal{O}_{\text{Jill}}$ satisfies this condition of simultaneity. When $\tilde{\mathcal{O}}_{\text{Jack}}$ gets to the end of a period L, the surface of simultaneity through the origin does not match up. During the course of his adventure there are points connected by timelike lines on this surface, which itself violate the concept of Lorentzian geometry[7].

Unwrapping the cylinder (Figure 14) provides us with a clear representation of this asymmetry, which ends up being a common theme for solving such paradoxes, and provides us insight into how we can break the solution once more to create the next

[6] The *elapsed proper time* along any path as measured by a clock carried along the path is the Lorentzian length of the path. [37]

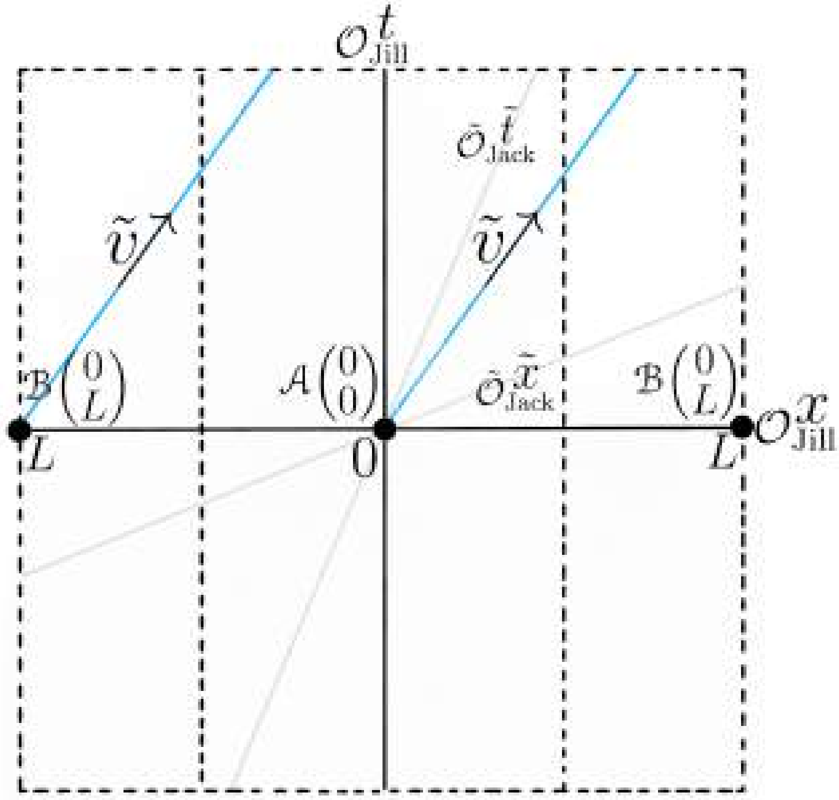
[7] In Lorentzian geometry the direct (timelike) path between two points always takes *more* time than any indirect (timelike) path. [37]

paradox to solve. In some respects I feel like this is a *proof by destruction* as you will, where you can remove a domino and show the domino effect still applies!

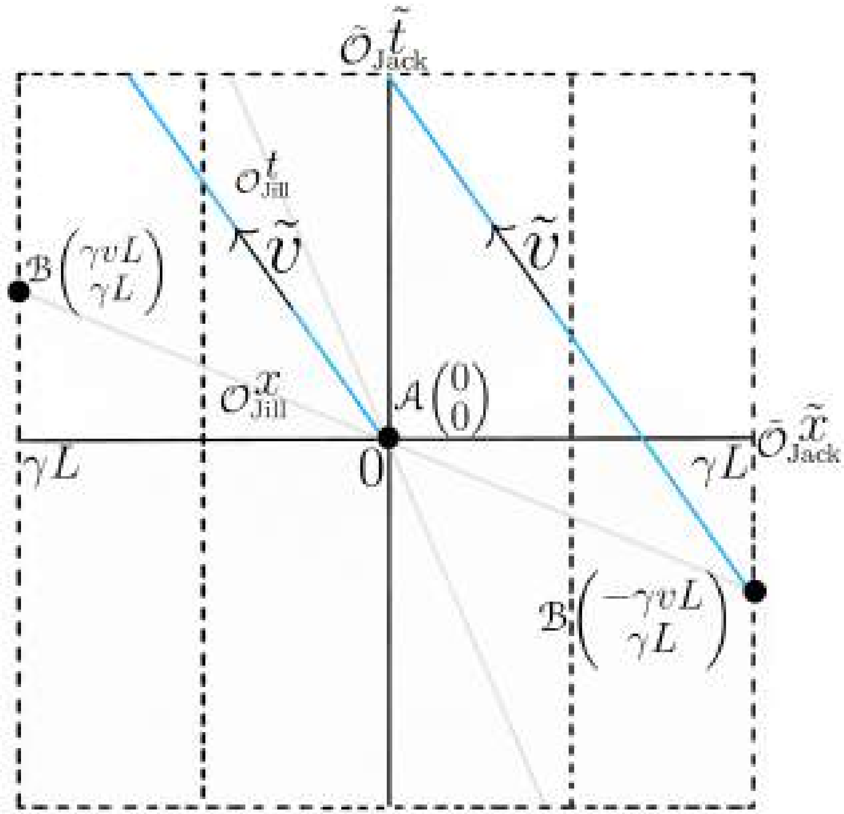
Graphically we can say that at $\tilde{\mathcal{A}}\binom{0}{0}$ we have one copy of $\tilde{\mathcal{O}}_{\text{Jack}}$ residing at home with $\mathcal{O}_{\text{Jill}}$, with another version of $\tilde{\mathcal{O}}_{\text{Jack}}$ at $\mathcal{B}\begin{pmatrix}-\gamma vL \\ \gamma L\end{pmatrix}$. From $\mathcal{O}_{\text{Jill}}$'s perspective each version of $\tilde{\mathcal{O}}_{\text{Jack}}$ leave simultaneously. But from $\tilde{\mathcal{O}}_{\text{Jack}}$'s perspective departures happen sequentially, illustrating the asymmetry. On each pass $\tilde{\mathcal{O}}_{\text{Jack}}$'s clock will be slower and slower when compared to $\mathcal{O}_{\text{Jill}}$'s, and they will never have the opportunity to meet again in space-time.

In conclusion, we note that it is not sufficient for two observers to agree on elapsed time, and rely on the symmetry of motion. Worldlines must also be symmetric between frames in space-time for agreement. Philosophically, this means that both $\tilde{\mathcal{O}}_{\text{Jack}}$ and $\mathcal{O}_{\text{Jill}}$ have to perceive the universe in the same way to agree on elapsed time, which in itself is quite the conceptual argument!

In Figure 14 we show the situation as translated from both perespectives. [24, 26, 37]



(a) $\mathcal{O}_{\text{Jill}}$'s rest frame.



(b) $\tilde{\mathcal{O}}_{\text{Jack}}$'s inertial frame.

Figure 14: In $\mathcal{O}_{\text{Jill}}$'s frame departures are simultaneous, but not in $\tilde{\mathcal{O}}_{\text{Jack}}$'s.

3.6 Shining a Torch in the Dark

Suppose that $\tilde{\mathcal{O}}_{\text{Jack}}$, in his inertial spaceship sends light rays in both directions around the universe. When does $\tilde{\mathcal{O}}_{\text{Jack}}$, by his own clock, recapture these light rays? What can $\tilde{\mathcal{O}}_{\text{Jack}}$ infer about his own velocity and period L from such measurements?

Previously, we demonstrated in section 3.4.3 and figure 14, that $\tilde{\mathcal{O}}_{\text{Jack}}$'s inertial frame is asymmetric due to his inability to synchronise clocks. In $\mathcal{O}_{\text{Jill}}$'s frame, light rays emitted in opposite directions return to the same point at the same time after each cycle. Light rays recaptured in $\tilde{\mathcal{O}}_{\text{Jack}}$'s frame to the right take $\tilde{\tau}_+ = \frac{\gamma L}{1+v}$ to make one cycle, yet rays to the left take $\tilde{\tau}_- = \frac{\gamma L}{1-v}$ as described in Peters paper [26]. If we take the ratio of these two measurements to eliminate L we arrive at

$$\begin{aligned}\frac{\tilde{\tau}_-}{\tilde{\tau}_+} &= \frac{\gamma L}{1+v}\bigg/\frac{\gamma L}{1-v} \\ &= \frac{1-v}{1+v}\end{aligned} \tag{3.4}$$

where v is shown as $\tilde{v}$ in Figure 15b. We aim to reproduce this ratio in the preferred frame of $\mathcal{O}_{\text{Jill}}$, and further demonstrate the ability to infer L and v from this.

As the difference of times depends on two variables v and L, he can attempt to draw conclusions on how many cycles he has previously made around the universe, provided he can accurately measure his velocity and elapsed time. However, we have previously clarified that he is unable to synchronise his clocks accurately, and this discontinuity prevents him from doing so.

We therefore consider $\mathcal{O}_{\text{Jill}}$'s frame to simplify our calculations, and arrive at formulas for the proper time from the left and right incident rays meeting with $\tilde{\mathcal{O}}_{\text{Jack}}$ shown in Figure 16. $\tilde{\mathcal{O}}_{\text{Jack}}$'s worldline is represented by the line

$$x = \tilde{v}t. \tag{3.5}$$

The worldlines for the two lightrays are represented by c_+

$$x = t - L \tag{3.6}$$

and c_-

$$x = -t + L. \tag{3.7}$$

Thus the coordinates of event $\mathcal{A}$ incident with lightray c_- can be found via equating

$$\begin{aligned}
\tilde{v}t &= -t + L \\
t(\tilde{v} + 1) &= L \\
t &= \frac{L}{\tilde{v} + 1} \\
x &= -\frac{L}{\tilde{v} + 1} + L \\
&= \frac{\tilde{v}L}{\tilde{v} + 1}.
\end{aligned}$$

Event $\mathcal{A}$ is then represented by the coordinate

$$\mathcal{A}\begin{pmatrix} \frac{L}{\tilde{v}+1} \\ \frac{\tilde{v}L}{\tilde{v}+1} \end{pmatrix}. \tag{3.8}$$

The coordinates of event $\mathcal{B}$ incident with lightray c_+ can be found via equating

$$\begin{aligned}
\tilde{v}t &= t - L \\
t(\tilde{v} - 1) &= -L \\
t &= \frac{L}{1 - \tilde{v}} \\
x &= \frac{L}{1 - \tilde{v}} - L \\
&= \frac{\tilde{v}L}{1 - \tilde{v}}.
\end{aligned}$$

Event $\mathcal{B}$ is then represented by the coordinate

$$\mathcal{B}\begin{pmatrix} \dfrac{L}{1-\tilde{v}} \\ \dfrac{\tilde{v}L}{1-\tilde{v}} \end{pmatrix}. \tag{3.9}$$

The proper time measured by $\mathcal{O}_{\text{Jill}}$ is defined as

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2\,, \tag{3.10}$$

therefore the proper time of the c_+ lightray recaptured in Figure 16 at event $\mathcal{B}$ is

$$\begin{aligned} (\Delta\tau_{\mathcal{B},+})^2 &= \left(\frac{L}{1-\tilde{v}}\right)^2 - \left(\frac{\tilde{v}L}{1-\tilde{v}}\right)^2 \\ &= \frac{L^2}{(1-\tilde{v})^2} - \frac{\tilde{v}^2L^2}{(1-\tilde{v})^2} \\ &= \frac{L^2(1-\tilde{v}^2}{(1-\tilde{v}^2)} \\ \therefore \tau_{\mathcal{B},+} &= \frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}}, \end{aligned} \tag{3.11}$$

and the proper time of the c_- lightray recaptured in Figure 16 at event $\mathcal{A}$ is

$$\begin{aligned} (\Delta\tau_{\mathcal{A},-})^2 &= \left(\frac{L}{\tilde{v}+1}\right)^2 - \left(\frac{\tilde{v}L}{\tilde{v}+1}\right)^2 \\ &= \frac{L^2}{(\tilde{v}+1)^2} - \frac{\tilde{v}L^2}{(\tilde{v}+1)^2} \\ &= \frac{L^2(1-\tilde{v}^2)}{(\tilde{v}+1)^2} \\ \therefore \tau_{\mathcal{A},-} &= \frac{L\sqrt{1-\tilde{v}^2}}{1+\tilde{v}}. \end{aligned} \tag{3.12}$$

We have two formulas relating $\tilde{v}$ and L to which $\tilde{\mathcal{O}}_{\text{Jack}}$ can begin to understand his period and velocity in the universe. Event $\mathcal{B}$ represents the recaptured c_+ lightray with proper time $\tau_{\mathcal{B},+} = \frac{L}{\gamma(1-\tilde{v})}$ shown in equation (3.11). Event $\mathcal{A}$ represents the recaptured c_- lightray with proper time $\tau_{\mathcal{A},-} = \frac{L}{\gamma(1+\tilde{v})}$ shown in equation (3.12).

If we define the ratio between the two measured proper times to be

$$\frac{\tau_{\mathcal{A},-}}{\tau_{\mathcal{B},+}} = \hat{\tau}, \tag{3.13}$$

we arrive at

$$\begin{aligned} \hat{\tau} &= \frac{L\sqrt{1-\tilde{v}^2}}{1+\tilde{v}} \Bigg/ \frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}} \\ &= \frac{1-\tilde{v}}{1+\tilde{v}}. \end{aligned} \tag{3.14}$$

Rearranging to make the velocity $\tilde{v}$ the subject gives us

$$\begin{aligned} (\tilde{v}+1)\hat{\tau} &= 1-\tilde{v} \\ \tilde{v}(\hat{\tau}+1) &= 1-\hat{\tau} \\ \tilde{v} &= \frac{1-\hat{\tau}}{1+\hat{\tau}}. \end{aligned} \tag{3.15}$$

Substituting this expression into $\tau_{\mathcal{A},-}$ we can also rearrange to make L the subject. From equation (3.11) and (3.12) we have

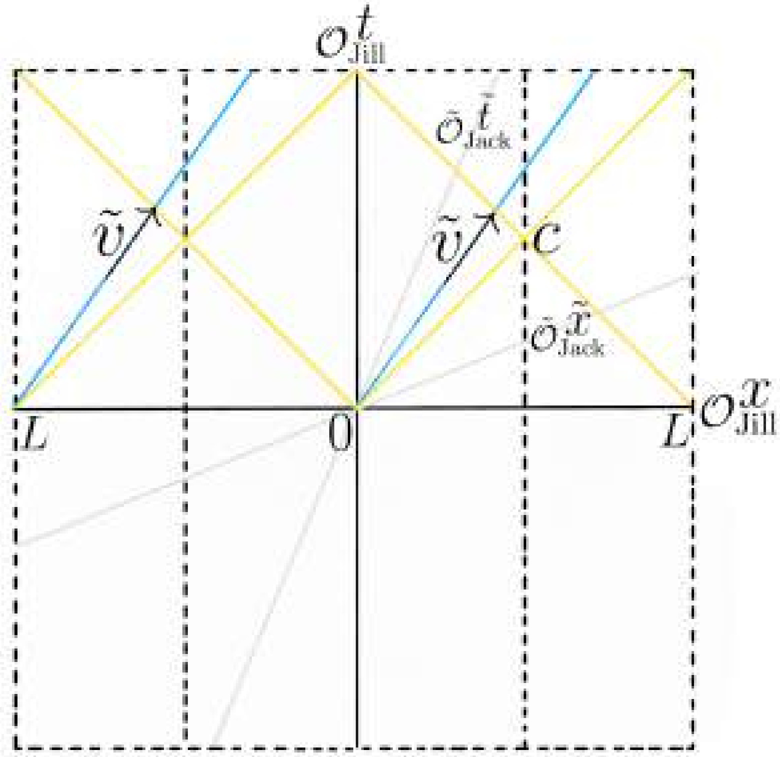
$$\begin{aligned} \tau_{\mathcal{A},-} &= \frac{L\sqrt{1-\tilde{v}^2}}{1+\tilde{v}}, \\ \tau_{\mathcal{B},+} &= \frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}} \end{aligned}$$

which then follows that

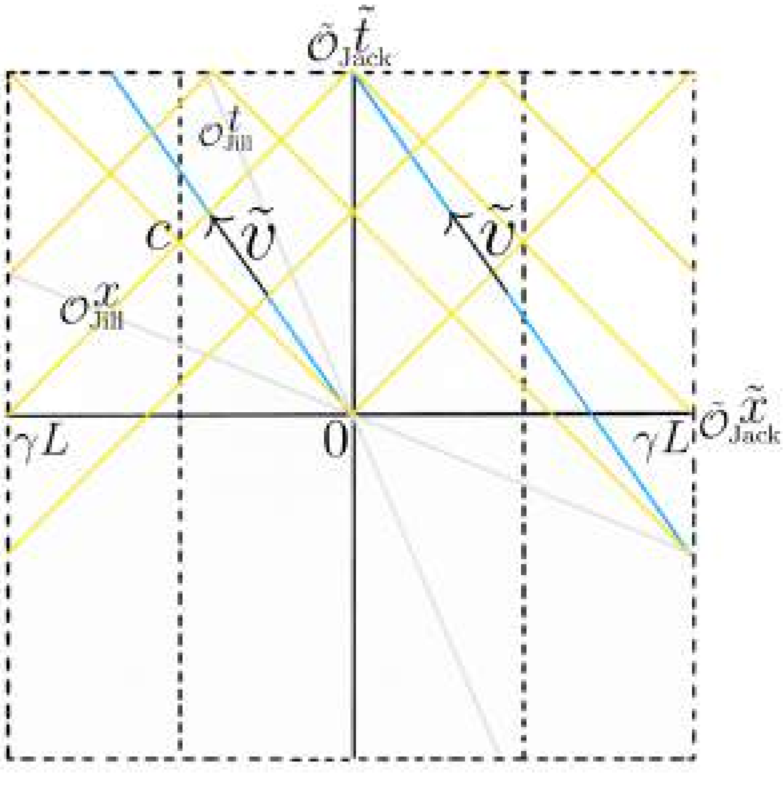
$$\begin{aligned} \tau_{\mathcal{A},-}\tau_{\mathcal{B},+} &= \left(\frac{L\sqrt{1-\tilde{v}^2}}{1+\tilde{v}}\right)\left(\frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}}\right) \\ &= \frac{L^2(1-\tilde{v}^2)}{(1-\tilde{v}^2)} \\ \therefore L &= \sqrt{\tau_{\mathcal{A},-}\tau_{\mathcal{B},+}}. \end{aligned} \tag{3.16}$$

Upon recapturing both lightrays c_+ and c_- at events $\mathcal{B}$ and $\mathcal{A}$ respectively, we arrive at the conclusion that if the adventurer was to make a note of the proper times at both events, he would be able to infer his velocity and period from such a journey. The important results from this analysis are then the formulas

$$\begin{aligned}\hat{\tau} &= \frac{\tau_{\mathcal{A},-}}{\tau_{\mathcal{B},+}} \\ L &= \sqrt{(\tau_{\mathcal{A},-})(\tau_{\mathcal{B},+})} \\ \tilde{v} &= \frac{1-\hat{\tau}}{1+\hat{\tau}} = \frac{(\tau_{\mathcal{B},+})-(\tau_{\mathcal{A},-})}{(\tau_{\mathcal{B},+})+(\tau_{\mathcal{A},-})}.\end{aligned} \tag{3.17}$$



(a) $\mathcal{O}_{\text{Jill}}$'s rest frame.



(b) $\tilde{\mathcal{O}}_{\text{Jack}}$'s inertial frame.

Figure 15: **(a)** $\mathcal{O}_{\text{Jill}}$ sees $\tilde{\mathcal{O}}_{\text{Jack}}$'s emitted lightrays depart simultaneously and is able to synchronise clocks to her left and right. **(b)** $\tilde{\mathcal{O}}_{\text{Jack}}$ emits lightrays initially and upon returning each cycle and finds himself unable to synchronise his clocks when comparing an emitted beam around the left of the cylinder to the right.

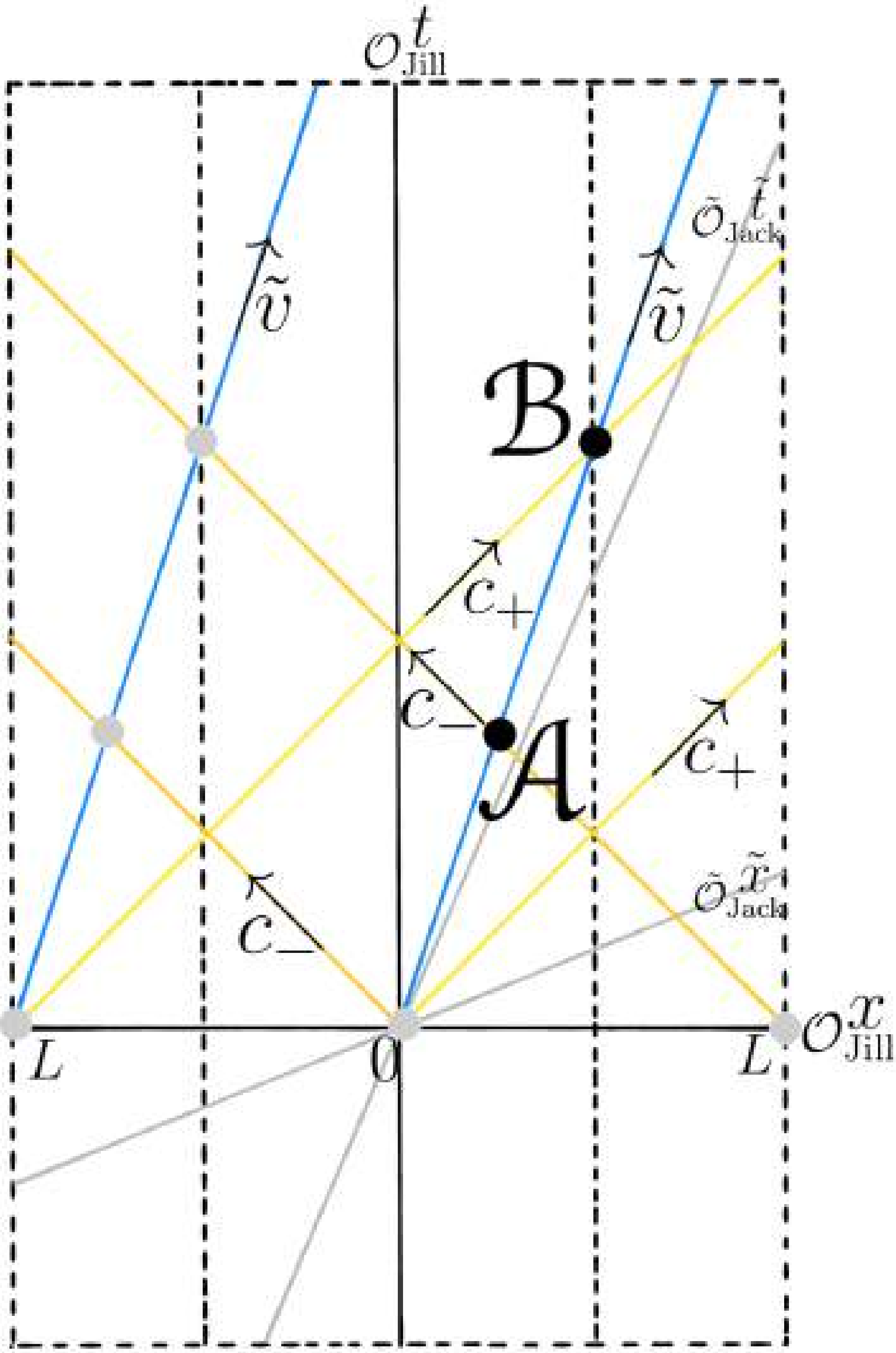


Figure 16: The preferred observer $\mathcal{O}_{\text{Jill}}$ receives light signals at the same point in spacetime. She notes that the light ray to the left collides with $\tilde{\mathcal{O}}_{\text{Jack}}$ before the light ray fired to the right. Due to the existence of this preferred frame, we choose to do calculations here as the result is the same in both frames. The two events displayed in black are the incident rays which meet $\tilde{\mathcal{O}}_{\text{Jack}}$ first respectively. From this we next show how upon comparison of these two measurements of proper time, he finds equations to determine his period L and velocity v.

3.7 Both Twins as Inertial Adventurers

Suppose that $\mathcal{O}'_{\text{Jill}}$ climbs in a spaceship, accelerates, and then turns her engines off. If $\mathcal{O}''_{\text{Jack}}$ and $\mathcal{O}'_{\text{Jill}}$ have different velocities, they pass each other from time to time. What can $\mathcal{O}''_{\text{Jack}}$ and $\mathcal{O}'_{\text{Jill}}$ infer about their velocities and period L from their clock readings as the time passes?

As $\mathcal{O}'_{\text{Jill}}$ has remained stationary for all other scenarios, in a rush to adventure, she sets off at a velocity close to the speed of light. $\mathcal{O}''_{\text{Jack}}$, in his exhaustion from earlier travels, slows down to have a look what she is up to, shown in Figure 18 from the rest frame of the universe we define as $\mathcal{O}_{\text{Universe}}$.

Firstly we aim to find the coordinates of $\mathcal{A}$ and $\mathcal{B}$ as before in Figure 18. The worldlines for coordinate $\mathcal{A}$ consist of $\mathcal{O}''_{\text{Jack}}$

$$x = -v''t \tag{3.18}$$

and $\mathcal{O}'_{\text{Jill}}$

$$x = v't - L. \tag{3.19}$$

We then find the coordinates in the usual way

$$\begin{aligned} -v''t &= v't - L \\ L &= v't + v''t \\ L &= t(v' + v'') \\ t &= \frac{L}{v' + v''} \\ x &= -\frac{v''L}{v' + v''}, \end{aligned}$$

which then follows that event $\mathcal{A}$ has coordinates

$$\mathcal{A}\begin{pmatrix} \dfrac{L}{v' + v''} \\ \dfrac{-v''L}{v' + v''} \end{pmatrix}. \tag{3.20}$$

Next, coordinates of event $\mathcal{B}$ consist of the worldlines of $\mathcal{O}''_{\text{Jack}}$

$$x = -v''t + L \tag{3.21}$$

and $\mathcal{O}'_{\text{Jill}}$

$$x = v't - L \tag{3.22}$$

where we arrive through the same method as before

$$\begin{aligned}
-v''t + L &= v't - L \\
2L &= v't + v''t \\
2L &= t(v' + v'') \\
t &= \frac{2L}{v' + v''} \\
x &= -\frac{2v''L}{v' + v''} + L \\
x &= \frac{L(v' - v'')}{v' + v''}
\end{aligned}$$

to find coordinate $\mathcal{B}$ to be

$$\mathcal{B}\begin{pmatrix} \dfrac{2L}{v' + v''} \\ \dfrac{L(v' - v'')}{v' + v''} \end{pmatrix}. \tag{3.23}$$

The proper time that $\mathcal{O}''_{\text{Jack}}$ measures between event $\mathcal{A}$ (Equation 3.21) and $\binom{0}{0}$ is

$$\begin{aligned}
(\Delta\tau_{\mathcal{A},\text{Jack}})^2 &= \left(\frac{L}{v' + v''}\right)^2 - \left(-\frac{v''L}{v' + v''}\right)^2 \\
&= \frac{L^2}{(v' + v'')^2} - \frac{(v''L)^2}{(v' + v'')^2} \\
&= \frac{L^2\left(1 - (v'')^2\right)}{(v' + v'')^2} \\
\therefore \tau_{\mathcal{A},\text{Jack}} &= \frac{L\sqrt{1 - (v'')^2}}{v' + v''}
\end{aligned} \tag{3.24}$$

which represents the proper time between his two successive meetings with $\mathcal{O}'_{\text{Jill}}$. The proper time that $\mathcal{O}'_{\text{Jill}}$ measures between event $\mathcal{A}$ (Equation 3.21) and $\begin{pmatrix} 0 \\ -L \end{pmatrix}$ is

$$\begin{aligned}
(\Delta\tau_{\mathcal{A},\text{Jill}})^2 &= \left(\frac{L}{v'+v''}\right)^2 - \left(-\frac{v''L}{v'+v''}+L\right)^2 \\
&= \frac{L^2\left(1-(v')^2\right)}{(v'+v'')^2} \\
\therefore\ \tau_{\mathcal{A},\text{Jill}} &= \frac{L\sqrt{1-(v')^2}}{v'+v''}
\end{aligned} \tag{3.25}$$

which represents the proper time between her two successive meetings with $\mathcal{O}''_{\text{Jack}}$. The Lorentz transformations we derived in the equation set (2.7) includes the velocity addition formulas

$$\frac{\Delta x'}{\Delta t'} = \tilde{u} = \frac{u-v}{1-uv} \tag{3.26}$$

and

$$\frac{\Delta x}{\Delta t} = u = \frac{\tilde{u}+v}{1+\tilde{u}v}, \tag{3.27}$$

which we can use to find another relationship to describe the velocities of $\mathcal{O}''_{\text{Jack}}$ and $\mathcal{O}'_{\text{Jill}}$ from the reference frame of the $\mathcal{O}_{\text{Universe}}$. $\mathcal{O}''_{\text{Jack}}$'s velocity is $-v''$ and $\mathcal{O}'_{\text{Jill}}$'s velocity is v', thus the velocity addition formula gives us

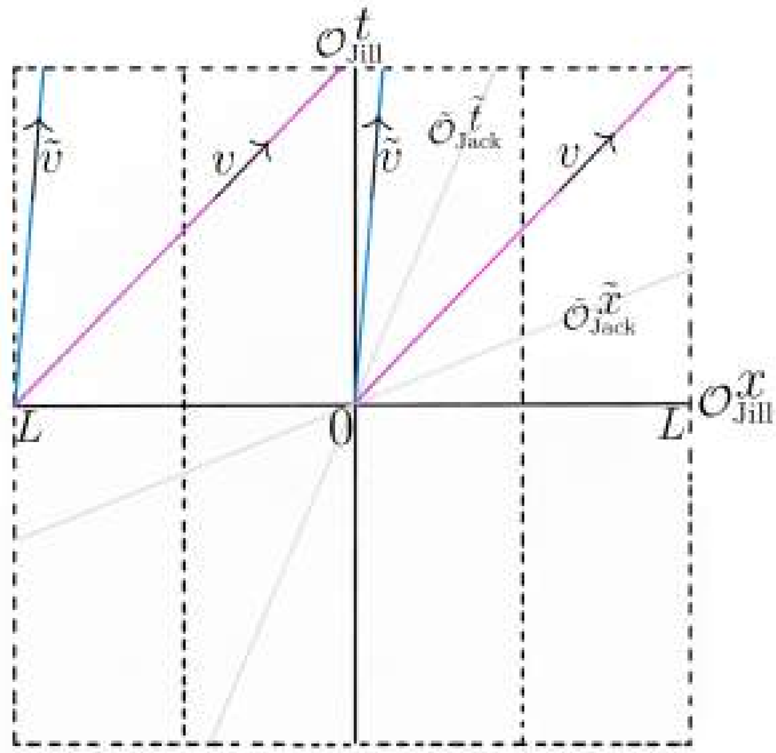
$$v = \frac{v'+v''}{1+v'v''}. \tag{3.28}$$

We then have the velocity v, which can be measured with a standard issue traffic control speed gun, and the two proper times as measured by $\mathcal{O}''_{\text{Jack}}$ and $\mathcal{O}'_{\text{Jill}}$. Our three measurable quantities as equations to solve then are

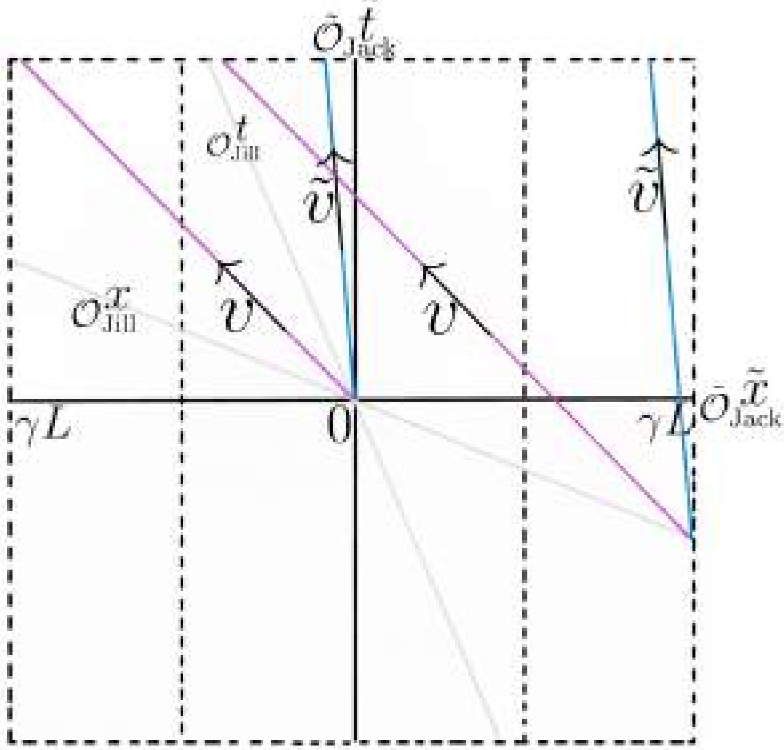
$$\begin{aligned}
v &= \frac{v'+v''}{1+v'v''} \\
\tau_{\mathcal{A},\text{Jill}} &= \frac{L\sqrt{1-(v')^2}}{v'+v''} \\
\tau_{\mathcal{A},\text{Jack}} &= \frac{L\sqrt{1-(v'')^2}}{v'+v''}.
\end{aligned} \tag{3.29}$$

The matlab code in appendix C finds that the solutions are

$$\begin{aligned} v' &= \frac{\tau_{\mathcal{A},\text{Jack}} + \tau_{\mathcal{A},\text{Jill}}\sqrt{1-v^2}}{v\tau_{\mathcal{A},\text{Jack}}} \\ v'' &= \frac{\tau_{\mathcal{A},\text{Jill}} + \tau_{\mathcal{A},\text{Jack}}\sqrt{1-v^2}}{v\tau_{\mathcal{A},\text{Jill}}} \\ L &= \sqrt{\frac{v^2\tau_{\mathcal{A},\text{Jill}}^2 - \tau_{\mathcal{A},\text{Jill}}^2 + v^2\tau_{\mathcal{A},\text{Jack}}^2 - \tau_{\mathcal{A},\text{Jack}}^2 - 2\tau_{\mathcal{A},\text{Jack}}\tau_{\mathcal{A},\text{Jill}}\sqrt{1-v^2}}{1-v^2}}. \end{aligned} \tag{3.30}$$



(a) $\mathcal{O}_{\mathrm{Jill}}$'s inertial frame.



(b) $\tilde{\mathcal{O}}_{\mathrm{Jack}}$'s inertial frame.

Figure 17: In compliment to Figure 19 this shows the worldlines of **(a)** $\mathcal{O}_{\mathrm{Jill}}$'s inertial frame and **(b)** $\tilde{\mathcal{O}}_{\mathrm{Jack}}$'s inertial frame.

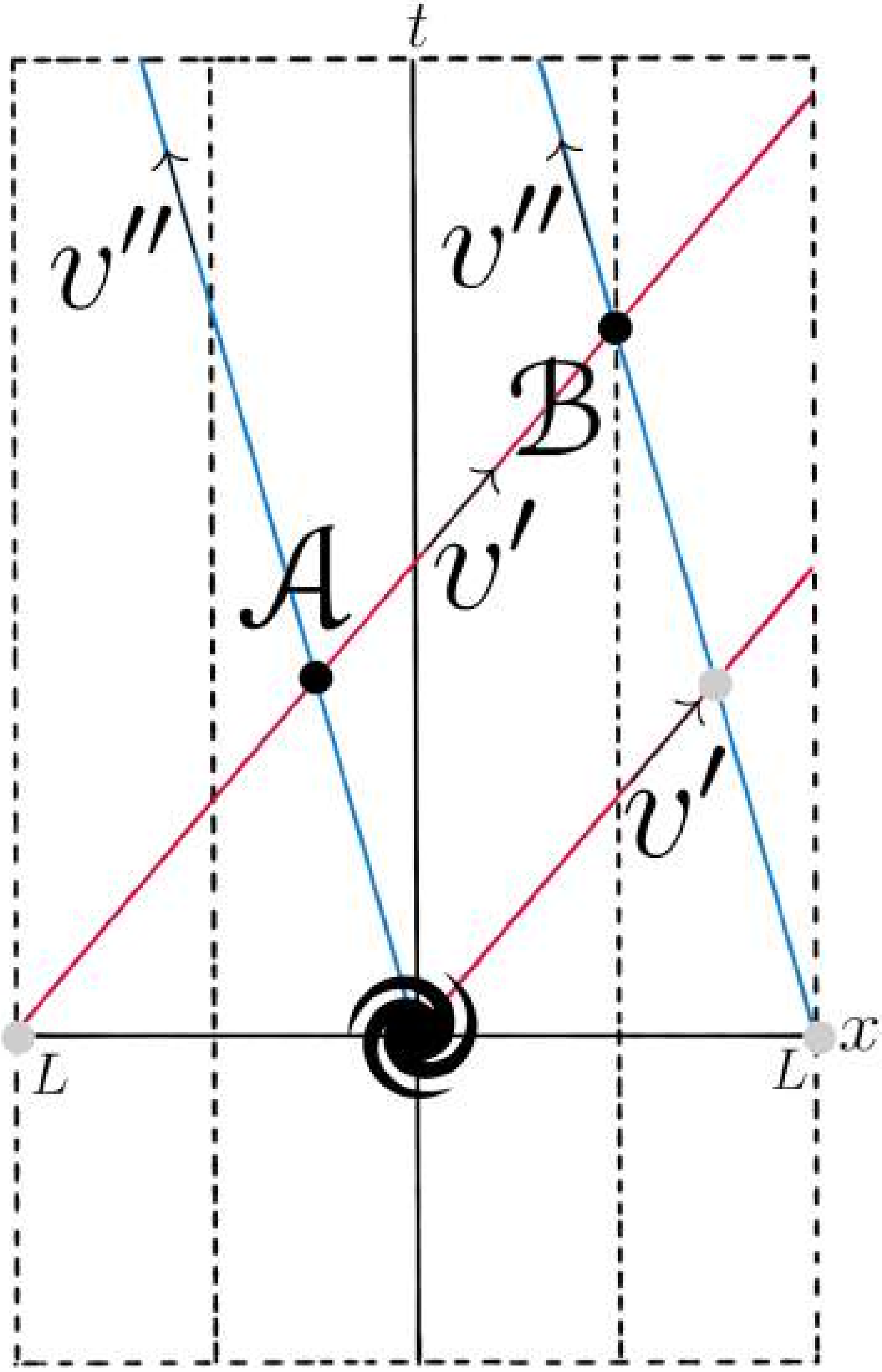


Figure 18: The third scenario shown here is from the rest frame of universe.

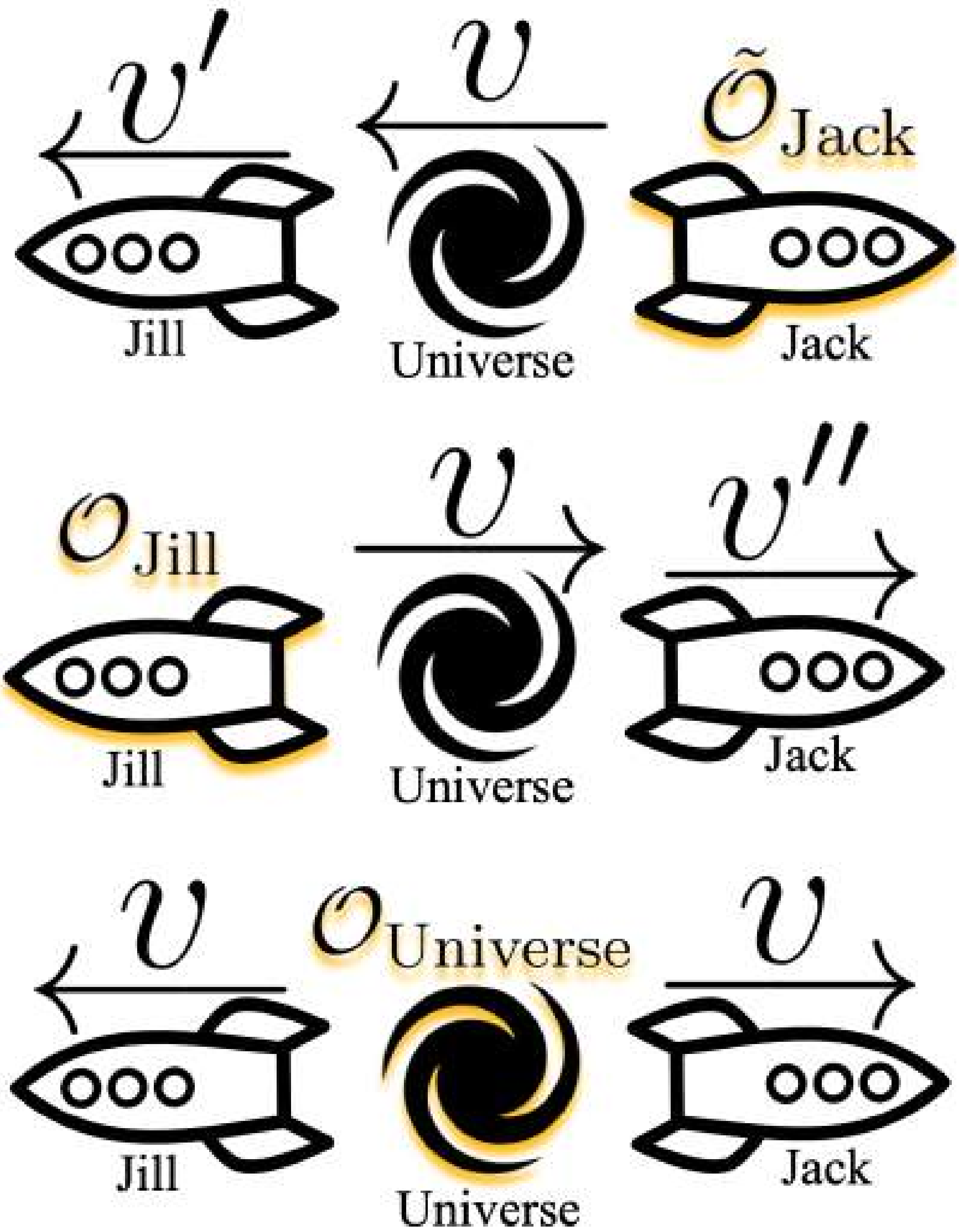


Figure 19: Shown from all 3 perspectives the velocities v indicated in the universe rest frame exhibit a special "neutral" velocity such that both observers move away at the same velocity.

4 Periodic Spacetime in Extra Dimensions

4.1 Overview of Topology

As stated in the introduction, topology is an extension of geometry which describes space, illustrating its features, dimensions, connectivity, orientability, infiniteness or finiteness, without measurement. Continuity is important in such a space, and can be visioned such that space can stretch without tearing and ends be glued together. Space can be stretched, crushed or kneaded in any way without changing its topology. We are primarily concerned with space of zero curvature so that special relativity still applies. Our spacetime occupies the infinite Euclidean plane and four others can be constructed from this.

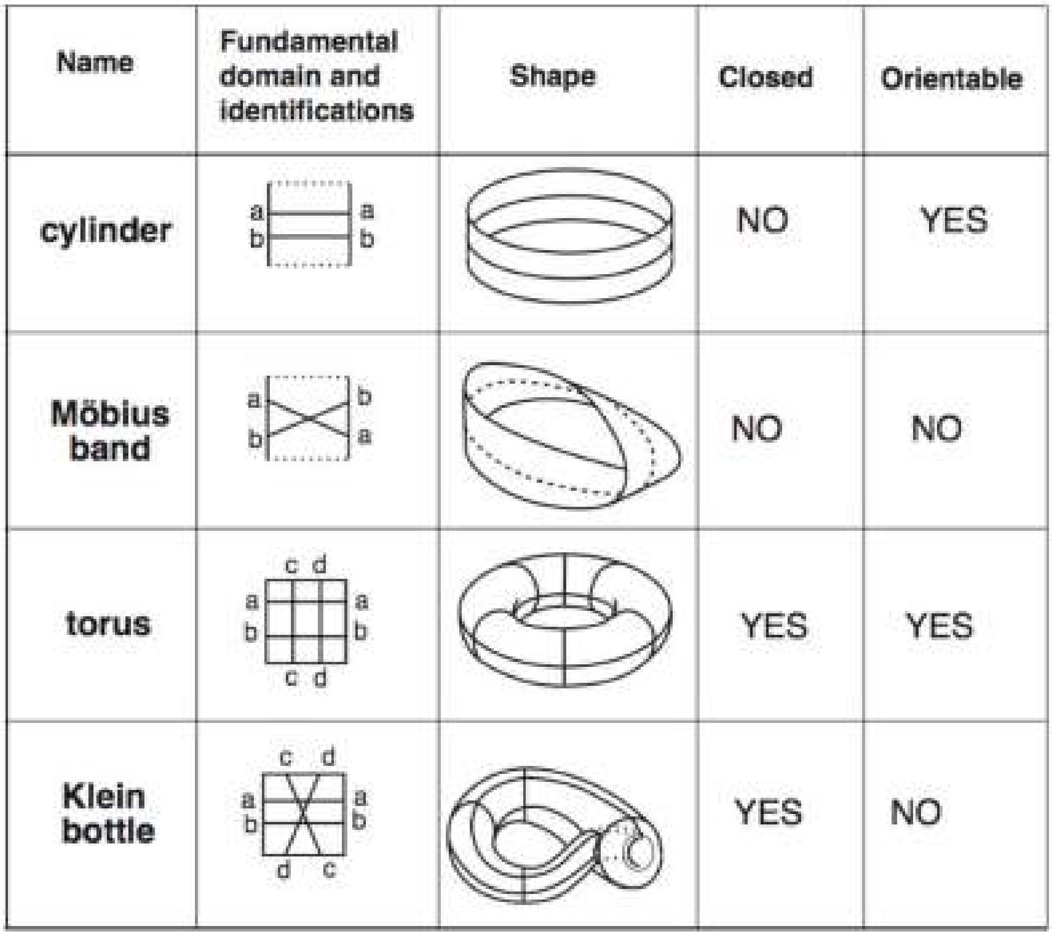


Figure 20: The four multiply connected topologies of the 2-dimensional Euclidean plane. [25]

Figure 20 shows how we can begin with a sheet of paper[8] and form a cylinder via gluing two sides of an infinite strip together, and a Möbius band by twisting an edge 180° before the same gluing process. The torus is then created by taking the other parallel sides and gluing those together, and the Klein bottle by twisting a pair of edges before gluing. The two that we will explore will be the $(2+1)D$ torus and the $(3+1)D$ hypertorus.

4.2 Constructing the $3D$ Torus

The fundamental domain of the Torus is the same sheet of paper previously used to construct the cylinder, which for clarity will now carry a subscript $\tilde{\mathcal{M}}_{\mathrm{C}}$. We can generate the periodic space in the same way, and illustrate it by unravelling the Torus into the Euclidean plane $\mathbb{R}^2$, illustrated below. This mapping can be considered as

$$\begin{pmatrix} t \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} t \\ x + L_x \\ y + L_y \end{pmatrix},$$

where previously we dealt with the spacetime $\begin{pmatrix} t \\ x \end{pmatrix}$ of the cylinder $\tilde{\mathcal{M}}_{\mathrm{C}}$. As an abbreviation, the torus in Minkowski spacetime will be defined as $\tilde{\mathcal{M}}_{\mathrm{T}}$.

4.3 The Periodic Twin Paradox in a $3D$ Torus

As before, in $\tilde{\mathcal{M}}_{\mathrm{T}}$, $\tilde{\mathcal{O}}_{\mathrm{Jack}}$ can remain in an inertial reference frame, provided he does not change direction, and returns periodically back to $\mathcal{O}_{\mathrm{Jill}}$. We have as before a similar scenario, where both will believe that the other is younger upon each pass. The resolution is still as before, due to the asymmetry between the spacetime paths of events, which in the standard paradox is caused by a change of acceleration. In a compact spacetime this asymmetry arises from multiply connected topology, and introduces the idea of preferred reference frames. The subtle difference however, between $\tilde{\mathcal{M}}_{\mathrm{C}}$ and $\tilde{\mathcal{M}}_{\mathrm{T}}$, is that the torus is closed.

[8]A sheet of paper is merely a standard Euclidean plane in context.

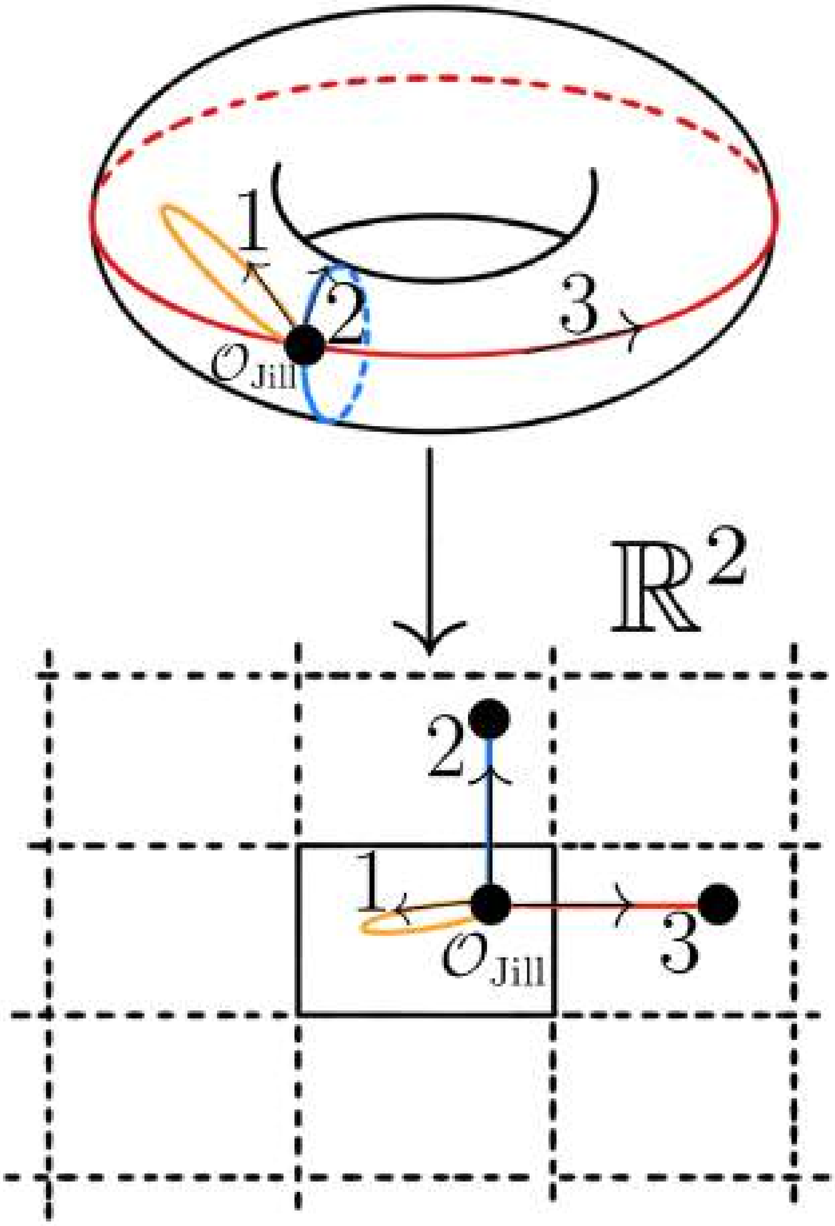


Figure 21: The paths 1, 2 and 3 all connect to $\mathcal{O}_{\text{Jill}}$. The closed curve 1 can be shrunk to a point, yet 2 and 3 cannot because they complete a cycle of spacetime and the asymmetry discovered in $\tilde{\mathcal{M}}_{\text{C}}$ for simultaneous departure presents itself again.

The classical twin paradox is illustrated via scenerio 1. As his worldline involved an acceleration he is in fact younger than $\mathcal{O}_{\text{Jill}}$. We can demonstrate the asymmetry discovered previously in this case by drawing closed curves on a surface. The closed curve can be tightened and reduced to a point without hitting an obstacle, or it cannot be tightened because it encounters a hole, such as $\tilde{\mathcal{M}}_{\text{C}}$ and $\tilde{\mathcal{M}}_{\text{T}}$. The prior is satisfied via the Euclidean plane, where surfaces are simply connected, and the latter case is a surface defined as being a multiply connected topology.

Two closed curves are *homotopic* if they can be repeatedly deformed into each other [25]. With this in mind we can define other classes of topologically equivalent closed curves. In Figure 22 the trajectories of both $\mathcal{O}_{\text{Jill}}$ and $\tilde{\mathcal{O}}_{\text{Jack}}$'s first scenario are homotopic to $\binom{0}{0}$, due to both trajectories being capable of deformation to a point. As before, they are asymmetric as only $\mathcal{O}_{\text{Jill}}$ stays in an inertial reference frame, where scenario 1 $\tilde{\mathcal{O}}_{\text{Jack}}$ undergoes a change of acceleration. Therefore among all homotopic curves from 0 to $0'$, we have but one which corresponds with an inertial observer, namely $\mathcal{O}_{\text{Jill}}$, and she experiences more time than $\tilde{\mathcal{O}}_{\text{Jack}}$ as before.

Scenario 2 $\tilde{\mathcal{O}}_{\text{Jack}}$ travels once along the "equator" of the torus, and scenario 3 $\tilde{\mathcal{O}}_{\text{Jack}}$ travels once around the circumference. Topologically, these journeys are not homotopic, as they can be characterised by a *winding index*. Our cylinder $\tilde{\mathcal{M}}_{\text{C}}$ has a winding index such that each period is counted as an integer upon each iteration. Naturally, from the previous construction, our torus $\tilde{\mathcal{M}}_{\text{T}}$ has a winding index consisting of $\binom{q}{p}$, where $q, p \in \mathbb{N}$. q represents the periodic count of the "equator", and p represents the periodic count of the circumference. Therefore $\mathcal{O}_{\text{Jill}}$ and scenario 1 $\tilde{\mathcal{O}}_{\text{Jack}}$ share the same winding index $\binom{0}{0}$, and scenarios 2 and 3 of $\tilde{\mathcal{O}}_{\text{Jack}}$ have respective winding indexes of $\binom{1}{0}$ and $\binom{0}{1}$. We have to conclude then that our asymmetry lies in that fact that the journeys of $\mathcal{O}_{\text{Jill}}$ and scenarios 2 and 3 of $\tilde{\mathcal{O}}_{\text{Jack}}$ belong to different homotopy classes. Thus the classical solution is the same, where $\mathcal{O}_{\text{Jill}}$ remains older than $\tilde{\mathcal{O}}_{\text{Jack}}$.

Generally then, for observers to record proper times and agree, we require more than just observers to be in inertial reference frames. Their worldlines must share the same homotopy class to agree on proper times. Regardless of any inertial observer, the oldest will always be the one whose worldline has a $\binom{0}{0}$ homotopy class. This further lends to the idea that preferred reference frames occur under such multi-connected compact spatial topology, such as $\mathcal{O}_{\text{Jill}}$'s frame in our example. This change of topology also infers a preferred homotopy class! [24, 25, 37]

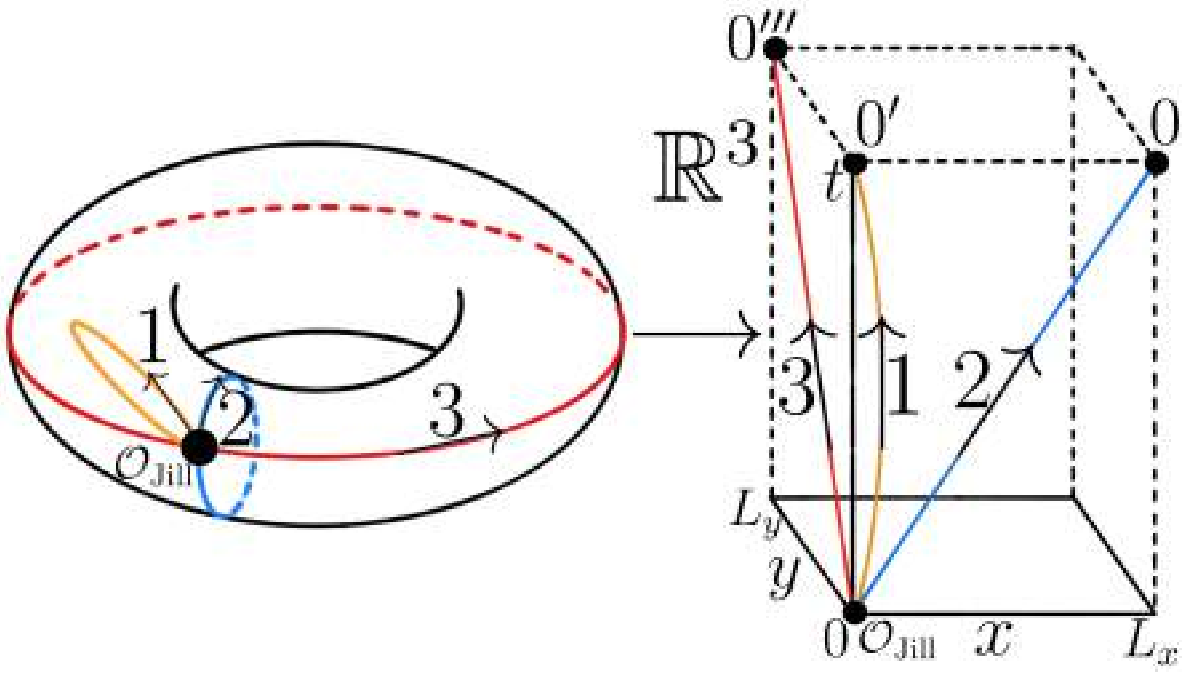


Figure 22: As before $\mathcal{O}_{\text{Jill}}$ stays at home and $\tilde{\mathcal{O}}_{\text{Jack}}$ makes 3 separate journeys in different scenarios here illustrated on one diagram. In scenario 1 he leaves home at $t = 0$ in his rocket, then turns back and returns home to $\mathcal{O}_{\text{Jill}}$. Scenarios 2 and 3 both involve travel from $t = 0$ to the respective points shown with non-accelerated worldlines. 2 around the circumference, and 3 along the "equator" of the torus.

4.4 Constructing the $4D$ Hypertorus

Here we will determine an alternate approach using the Lorentz transformations to arrive at the same conclusions found in $\tilde{\mathcal{M}}_{\text{C}}$ and $\tilde{\mathcal{M}}_{\text{T}}$. Previously, we have dealt with the spacetime $\begin{pmatrix} t \\ x \\ y \end{pmatrix}$, and we can of course generalise this to extra dimensions

$$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix},$$

where the flat spacetime metric is

$$\mathrm{d}s^2 = g_{\mu v}\mathrm{d}x^\mu \mathrm{d}x^v \text{ with } g^\mu_v = \mathrm{diag}(-1, 1, 1, 1). \tag{4.1}$$

This represents the manifold of special relativity where $\mathcal{R}\otimes\mathcal{M}$. $\mathcal{R}$ represents the direction of time and $\mathcal{M} = \mathbb{R}^3$ is a flat $3D$ infinite space. We can then define special relativity in a compact 3-manifold $\mathcal{M}_{\mathcal{C}} = \mathcal{R} \otimes \frac{\mathcal{M}}{\Gamma}$, where the group Γ is the set of instructions for introducing compact space. A period L in this context is described by the holonomies, $\phi \in \Gamma$, which maps the end point of one cycle to the origin. A periodic orbit is such that $x_{\text{end}} = \phi x_{\text{start}}$ where we define ϕ to be the composite word $\phi = \Pi_i^n \phi_{k_i}$. Next, if we embed our $(3+1)D$ spacetime to a $(4+1)D$ spacetime, we can introduce a fixed spatial coordinate q[9] of order unity to achieve

$$x^\alpha = \begin{pmatrix} t \\ x \\ y \\ z \\ q \end{pmatrix}.$$

As defined in section 2.5, Greek indices will be in the set $0, 1, 2, 3$ and Latin in $0, 1, 2, 3, 4$. To demonstrate a mapping from

$$\begin{pmatrix} t \\ x \\ y \\ z \\ q \end{pmatrix} \mapsto \begin{pmatrix} t \\ x + L_x \\ y + L_y \\ z + L_z \\ q \end{pmatrix}.$$

to map the end-point to the origin we can introduce 5×5 matrices as the elements of Γ, which generate this transition

$$T_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & L_x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ T_y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & L_y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ T_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & L_z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This generalises to

$$x^\alpha = \phi^a_b x^b \text{ for each } \phi \in \Gamma. \tag{4.2}$$

This hypertorus is constructed in a similar fashion to our $2D$ cylinder $\tilde{\mathcal{M}}$ by gluing opposite faces of the parallelepiped and has a periodic cycle of

$$x_{\text{end}} = T_y T_x^2 x_{\text{start}}.$$

[25, 28, 35]

[9]Embedding $3D$ spacetime into $4D$ allows us to write translations as matrices.

4.5 The Periodic Twin Paradox in a $4D$ Hypertorus

Suppose that we have the same situation as described from before in $2D$, and apply to our newly defined general coordinates. As $\mathcal{O}_{\text{Jill}}$ is at rest the holonomies result in $\phi = 1$. $\tilde{\mathcal{O}}_{\text{Jack}}$, in his inertial frame, exhibits a coordinate system defined by $\tilde{x} = \Lambda x$, where Λ is the Lorentz transformation

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z & 0 \\ -\gamma v_x & 1 + \frac{(\gamma-1)v_x^2}{v^2} & \frac{(\gamma-1)v_x v_y}{v^2} & \frac{(\gamma-1)v_x v_z}{v^2} & 0 \\ -\gamma v_y & \frac{(\gamma-1)v_x v_y}{v^2} & 1 + \frac{(\gamma-1)v_y^2}{v^2} & \frac{(\gamma-1)v_y v_z}{v^2} & 0 \\ -\gamma v_z & \frac{(\gamma-1)v_x v_z}{v^2} & \frac{(\gamma-1)v_y v_z}{v^2} & 1 + \frac{(\gamma-1)v_z^2}{v^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and v_i are the velocities of boosts in $\begin{pmatrix} x \\ y \\ z \\ q \end{pmatrix}$, $v^2 = \sum_i v_i^2$, and $\gamma = \frac{1}{\sqrt{1-v^2}}$.

$\mathcal{O}_{\text{Jill}}$ sees $\tilde{\mathcal{O}}_{\text{Jack}}$ travel a distance d, and during one cycle she sees an elapsed time of

$$\Delta t = \frac{d}{v}.$$

She then determines $\tilde{\mathcal{O}}_{\text{Jack}}$'s clock to run slower by

$$\Delta \tilde{t} = \frac{\Delta t}{\gamma}$$

and believes that his clock will experience

$$\Delta \tilde{t} = \frac{d}{\gamma v}. \tag{4.3}$$

$\tilde{\mathcal{O}}_{\text{Jack}}$ is therefore younger upon completing one cycle and returning to $\mathcal{O}_{\text{Jill}}$. As $\tilde{\mathcal{O}}_{\text{Jack}}$ agrees that he is younger than $\mathcal{O}_{\text{Jill}}$ there is no paradox. According to $\tilde{\mathcal{O}}_{\text{Jack}}$, both spacetime points can be identified, yet this results in difficulties to synchronise clocks. As $\tilde{\mathcal{O}}_{\text{Jack}}$ must remain on a periodic orbit to remain inertial, we can define his journey for one cycle via the mapping from equation 4.2

$$\tilde{x} = \Lambda x \mapsto \Lambda \phi x.$$

The asymmetry here is given by

$$\delta \tilde{t} = -\gamma v^i (x_i - \phi_i^j x_j)$$

where $i = 1, 2, 3, 4$ and $v^i = \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix}$. $\tilde{\mathcal{O}}_{\text{Jack}}$ travels a distance measured by $\mathcal{O}_{\text{Jill}}$ of

$$d = \sqrt{(x_i - \phi_i^j x_j)(x^i - \phi_j i x^j)} = \sqrt{\frac{dv_i}{v}(x^i - \phi_j i x^j)}$$

where $x^i = \begin{pmatrix} x \\ y \\ z \\ q \end{pmatrix}$, and shows $\tilde{\mathcal{O}}_{\text{Jack}}$'s clocks to be desynchronised by

$$\delta \tilde{t} = -\gamma v d.$$

$\tilde{\mathcal{O}}_{\text{Jack}}$ sees $\mathcal{O}_{\text{Jill}}$ leave him in the opposite direction and return, covering a distance of γd, and discovers that his clock reads

$$\Delta \tilde{t} = \frac{d}{\gamma v} = \text{ equation (4.3)} \tag{4.4}$$

in agreement with $\mathcal{O}_{\text{Jill}}$'s clock reading before. This generalises the works of [10, 11, 25, 26, 28, 35].

5 Cosmological Spacetimes

The *Cosmological Principle* states that the universe is, on average, isotropic (the same in all dimensions) and homogeneous (the same at all points). Our location is not special, nor does one location take preference over another.

As discussed before, compact topology suggests that there is a preferred reference frame, so in a cosmological context this also suggests that there is a preferred galaxy to take measurements from. Such a universe is the one which is at the centre of everything! Consequently, the ability to synchronize clocks is reserved only for this preferred galaxy, one in which you can observe the smallest volume of the Universe. This clearly violates the cosmological principle and further lends to that fact that relativity is not valid globally in such compact spacetimes. [12]

5.1 The Topology of the Universe

A universe with a flat or open geometry is assumed to be infinite, yet as the speed of light is finite this is difficult to prove. We can consider instead that the universe has non-trivial topology. Geometry in our context describes the local shape of spacetime, yet topology describes these globally. General Relativity tells us that the properties of matter determine geometry, but does not infer anything of topology.

If you take a sheet of paper and join two sides to form a cylinder, as we did with our mapping from $\mathcal{M} \mapsto \tilde{\mathcal{M}}$ before, we have imposed boundary conditions such that an inertial traveller can adventure forever. However, displacement is finite due to the periodicity. If we then bend this cylinder to join its two ends together, we end up with a torus which is still a closed, finite $2D$ surface. This demonstrates how a torus can be constructed with a flat geometry. The surface may show signs of curvature but this is just a global representation of how it looks in $3D$ space. Locally it is impossible to distinguish between an infinite plane or the surface of a torus.

[10] A flat geometry such that the volume is finite and an inertial traveller would return to the starting point in a finite time.

The possibility then, is that our Universe could have a non-trivial topology[10], yet if the scale of this is much larger than the observable Universe, we would have no detection method to determine such. If the topology was smaller we would essentially see the periodicity occur, and evidence of a cycle would allow us to observe the beginning at the cosmological horizon. As we have no evidence of such topology observationally, the only scenarios left are that either this periodicity occurs beyond our limits of observation, or that we live in a place with standard topology. [12]

5.2 The Periodic Universe

From defining the homotopy class in section 4.3, we discovered topological invariance associated with each observers worldlines, which explained the asymmetry between inertial reference frames. This means that the rules of special relativity are being violated somehow. If we consider two assumptions we must abide by:

- Under special relativity two reference frames are invariant if the Lorentz transformation between them is also invariant.
- The Poincaré group contains the set of all Lorentz transformations consisting of a $10D$ group combining homogeneous Lorentz transformations consisting of translations, rotations and *boosts*.

The equivalence lost between inertial reference frames arises when introducing compact topology which fundamentally violates the Poincaré group.

However, Einstein's field equations in general relativity involve curved spacetimes with no specific symmetry, and our reasoning involved Lorentzian sections of spacetime. It is known that all exact solutions admit symmetry groups [6].

In cosmology, the big bang model, as modelled by the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{5.1}$$

with the metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right], \tag{5.2}$$

assumes the universe to be homogeneous and isotropic. Geometrically, this implies that spacelike slices have constant curvature. About each point, space is spherically symmetric. Group theory would describe this as spacetime invariance under a $6D$ isometry group. The curvature k of space can either be 0, positive or negative, corresponding to Euclidean space $\mathbb{R}^3$, the sphere $\mathbb{S}^3$ or the hyperbolic $\mathbb{H}^3$. Introducing compact space to these by gluing points together, lowers the dimension of their isometry group, yet the $3D$ homogeneity group is unaffected. This means that the isometry group is broken globally, meaning that preferred locations and directions exist such that the universe appears different.

We have to conclude that, in a Friedmann described universe, both the expansion of the universe, and the existence of multiply connected topology for constant time hypersurfaces, break Poincaré invariance. This singles out the same preferred inertial observer as before, who will age more quickly than her twin, comoving with the cosmic fluid.[6, 25, 31]

5.3 Compact Time

Previously, we have only considered implications of compactifying space, yet what can we say about imposing the same conditions on time? Any meaningful result would have to ensure causality violation did not occur along closed timelike curves. The scale of compact time chosen would be of interest here, and in terms of biological timescales, the scale of time would have to be much larger or shorter than the biological timescale, otherwise no new life could be created as it would have to somehow grow young again.

Cosmologically, the big bang could happen, followed by a crunch, only to repeat its cycle again ad infinitum [14]. What has happened, happening and yet to happen, will in turn happen again, lending itself to a cyclic time state. Such boundaries define fate to be a fundamental idea of physics in such a universe, yet as we understand from quantum gravity, upon each reset no two iterations would be the same. The initial conditions are slightly altered upon each iteration, allowing new galaxies, stars and life to exist.

If an omnipotent observer was the creator of it all, perhaps this is the variation by design to spare himself the relentless cosmic boredom of watching what has happened before, happen again. The nature of time is a concept elusive to define philosophically, and as we have shown is complicated to define scientifically. Introducing the idea that it could reset itself further illustrates its inescapable strangeness.

Part III

Resolving the Asymmetry

6 Conclusion

6.1 Why Twins?

Twin duality is ubiquitous throughout history, and such ideas stand the test of time as they invoke our primal nature. If we consider the twin paradox itself out of the context of special relativity, we in essence have a fable much like the stories of old. When we say twins in our scientific scenario, it is there simply to define them as both being the same age for experimental purposes. Yet the concept of twins can be that of a good versus evil, and as such pops up all over throughout mythology. Conceptually, the argument arises to that of a disagreement in perspective, which is another philosophical opinion based argument that is difficult to resolve.

For example, Romulus and Remus arrived in the area of seven hills, yet disagreed upon which hill to build. Romulus preferred the Palatine yet Remus preferred the Aventine. Ultimately this dispute led to Romulus killing Remus and then founded the city of Rome.

Or we could consider the sun god Ra and the god of chaos Apophis. Apophis was said to lie below the horizon line, waiting to devour Ra as Ra descended into the underworld. As he swallowed Ra, the sun would set, and upon swallowing led to nighttime. Eventually he would spit Ra back out, however and cause the sun to rise once more.

However, regardless of the story the theme is similar. An inability to agree from different perspectives, and a refusal to accept the underlying truth of reality which breaks our intuitive understanding. The twin paradox is exactly this, a paradox only until the subtleties relativity is truly understood.

6.2 Asymmetry and Preferred Frames

The initial paradox was solved simply from a lack of considering $\tilde{\mathcal{O}}_{\text{Jack}}$'s change of direction, and thus occupying a non-inertial reference frame during this transition. What if we could create a universe such that an adventurer could maintain constant acceleration indefinitely? To which we defined the cylinder, and saw it too had a resolution but a much more subtle one. Introducing a compact topology had its own consequences to the paradox, revealed as an asymmetry between frames. Even before illustrating the situation clearly via Minkowski diagrams, the mapping from

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} t \\ x + L \end{pmatrix}$$

results in a translation to another inertial observer

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{t} - v\gamma L \\ \tilde{x} + \gamma L \end{pmatrix},$$

and clearly shows that $\tilde{\mathcal{O}}$ has a time difference for departure upon each periodic interval. Whilst this provides us with a resolution to the paradox, what else can we draw from such a conclusion? As $\mathcal{O}$ is not hindered by this, we can conclude that calculations taken in this frame will be simpler to derive, leading us to the idea of a preferred frame of reference. Our most significant result from imposing periodic boundary conditions then, is the existence of this preferred frame.

The scenario we introduced in Shining a Torch in the Dark (Section 3.6), showed how we could arrive at formulas for $\tilde{\mathcal{O}}_{\text{Jack}}$'s velocity $\tilde{v}$ and period L in $\mathcal{O}_{\text{Jill}}$'s frame. This means that regardless of his place in the universe she is able to work out how many times he has traversed compact spacetime simply from comparing light rays incident from both directions around the cylinder. As we previously clarified, due to the existence of a preferred frame, we chose to do our calculations in $\mathcal{O}_{\text{Jill}}$'s frame to demonstrate we arrive at the same conclusions as Peters [26] considers in his S' frame, corresponding to $\tilde{\mathcal{O}}_{\text{Jack}}$'s frame in our example. The important results from this analysis were the formulas

$$\begin{aligned} \hat{\tau} &= \frac{\tau_{\mathcal{A},-}}{\tau_{\mathcal{B},+}} \\ L &= \sqrt{(\tau_{\mathcal{A},-})(\tau_{\mathcal{B},+})} \\ \tilde{v} &= \frac{1-\hat{\tau}}{1+\hat{\tau}} = \frac{(\tau_{\mathcal{B},+}) - (\tau_{\mathcal{A},-})}{(\tau_{\mathcal{B},+}) + (\tau_{\mathcal{A},-})} \end{aligned} \tag{6.1}$$

which are verified in appendix B. This meant that, to determine $\tilde{\mathcal{O}}_{\text{Jack}}$'s L and $\tilde{v}$ (Equation 6.1), $\mathcal{O}_{\text{Jill}}$ only needs to determine the two respective proper times $\tau_{\mathcal{A},-}$ and $\tau_{\mathcal{B},+}$. $\tilde{\mathcal{O}}_{\text{Jack}}$ from his frame however, is unable to accurately synchronise clocks in the same fashion and notes the discrepancy demonstrated in Peters paper [26] as

$$\tau_{\mathcal{B},+} - \tau_{\mathcal{A},-} = 2\gamma\tilde{v}L, \tag{6.2}$$

demonstrated graphically in Figure 15b.

The second scenario, Both Twins as Inertial Adventurers (Section 3.7), used a third inertial reference frame of a point in $\mathcal{O}_{\text{Universe}}$[11] to arrive at similar formulas to before in terms of the proper times measured, depending on L, v' and v''. To arrive at this $\mathcal{O}''_{\text{Jack}}$ fired a speed gun at $\mathcal{O}'_{\text{Jill}}$ to determine her speed which is defined as a measurable quantity v. The three measurable quantities as equations to solve were

$$\begin{aligned} v &= \frac{v' + v''}{1 + v'v''} \\ \tau_{\mathcal{A},\text{Jill}} &= \frac{L\sqrt{1-(v')^2}}{v' + v''} \\ \tau_{\mathcal{A},\text{Jack}} &= \frac{L\sqrt{1-(v'')^2}}{v' + v''} \end{aligned} \tag{6.3}$$

[11]Ultimately just any arbitrary point in the universe at rest.

and the matlab code in appendix C found the solutions to be

$$\begin{aligned}
v' &= \frac{\tau_{\mathcal{A},\text{Jack}} + \tau_{\mathcal{A},\text{Jill}}\sqrt{1-v^2}}{v\tau_{\mathcal{A},\text{Jack}}} \\
v'' &= \frac{\tau_{\mathcal{A},\text{Jill}} + \tau_{\mathcal{A},\text{Jack}}\sqrt{1-v^2}}{v\tau_{\mathcal{A},\text{Jill}}} \\
L &= \sqrt{\frac{v^2\tau_{\mathcal{A},\text{Jill}}^2 - \tau_{\mathcal{A},\text{Jill}}^2 + v^2\tau_{\mathcal{A},\text{Jack}}^2 - \tau_{\mathcal{A},\text{Jack}}^2 - 2\tau_{\mathcal{A},\text{Jack}}\tau_{\mathcal{A},\text{Jill}}\sqrt{1-v^2}}{1-v^2}}.
\end{aligned} \tag{6.4}$$

This tells us that we can determine v' v'' and L from our three measurable quantities alone.

To further illustrate the asymmetry and preferred frame results, we then constructed a torus from the same universal covering space (the sheet of paper) as before. Here we took an alternate approach and investigated a resolution via topology considerations, in which homotopy classes played a fundamental role. This allowed us to truly consider all possibilities an adventurer could make in such a spacetime, and revealed consequences to us which arose naturally in the transition to compactify space. We showed that each journey could be summarised by its trajectory around the universe, and that for asymmetry to not occur the respective homotopy classes for such had to be in agreement.

Lastly, we considered a comparison with our own topology and cosmology, to draw conclusions on the feasibility of such spacetimes constructed in $\tilde{\mathcal{M}}_{\mathcal{C}}$ and $\tilde{\mathcal{M}}_{\mathcal{T}}$. Relativity tells us information about what happens locally but nothing upon the global scale. If the topology of our universe is in fact non-trivial, the examples described in this paper could be a reality for us, and all the implications that follow. We could one day journey to distant stars, and see even further away a reflection of our home.

The visible universe is 98 billion light years across, yet only 13.8 billion years old [12]. Nothing travels faster than the speed of light. It would take you longer than the universe has existed, to reach most points in the visible universe, even if you could travel at near light speed. Yet even worse, this assumes that the universe is static, which its not. The universe isn't only expanding, the rate of expansion is accelerating. The size of the greater universe is estimated to be 250 times larger than the visible universe and 7 trillion light years across. The overwhelming majority of the universe can never be seen because it's growing faster than light can travel across it. Eventually all of the visible universe will be so far away that it's light will never reach us and the visible universe will be limited to our local cluster of galaxies.

So if we consider the same evolution over time for the periodic universe, accelerated expansion would suggest an overlap of objects completing cycles ever faster and faster. If our universe is in fact periodic, we would most certainly see evidence of this strange behaviour occurring. Expansion in this context is essentially the same however, as we have shown that the size of the universe increases in relation to velocity. Perhaps then, due to this similarity, it could be possible that in fact its impossible to ascertain what kind of spacetime we truly occupy!

6.3 Further Reading

For calculations involving acceleration to illustrate where asymmetry in non-inertial reference frames lie, Gamboa's [36] paper takes the twin paradox in this direction and argues that

> *"One should be careful in invoking different versions of the twin paradox to explain a point about the "original" version: in the end, each version is a different problem, and the explanation of one obviously cannot carry to another."*

I think in all cases a well defined problem in physics is important otherwise there is a loss in clarity and a resolution is vague at best.

Grøn [29] considers a cosmological spacetime, expanding on the twin paradox in Schwarzchild spacetime. Previously under circular motion they showed that the twin with a non-vanishing 4-acceleration is older than his brother at the reunion and argued that in spaces that are asymptotically Minkowskian there exists an absolute standard of rest determining which twin is oldest at the reunion. Which in our case is the idea of the preferred reference frame. He then shows that in the case of vertical motion in Schwarzchild spacetime the opposite is true. The twin with a non-vanishing 4-acceleration is younger. It would be interesting to see the effects of introducing a compact topology to his investigation, and is a potential direction my previous work could move into.

Blau [33] also considers the idea of introducing compact topology to avoid a change of acceleration to return home. His paper considers superluminal speeds on the same cylinder we constructed and notes the importance of how you define the transition points between complete cycles of the universe. He states that

> *"Many who have studied special relativity have pondered the question, can a person travelling faster than light take a round-trip journey and return earlier than they set off? The answer could go either way, but for any given journey all observers will agree on whether the return was earlier or later than the departure. If space is circular rather than linear, however, a superluminal traveler can return home without changing direction, and the issue of whether the return is unambiguously later or earlier than the departure needs to be reconsidered."*

I think it would be interesting to consider superluminal speeds in our two scenarios Shining a Torch in the Dark (Section 3.6) and Both Twins as Inertial Adventurers (Section 3.7). While the preferred reference frame will certainly play an important role in calculating the trivial time transition, interesting effects may occur in the $\tilde{\mathcal{O}}$ frame.

Other considerations include [15, 16, 17, 31, 32, 34], all of which are variations on the theme of compact spacetime and the twin paradox.

Part IV

Appendix

A Definitions

A **Minkowski spacetime isometry** has the property that the interval between events is left invariant. A time or space reversal (a reflection) is also an isometry of this group. In **Minkowski space** (i.e. ignoring the effects of gravity), there are ten degrees of freedom of the **isometries**, which may be thought of as translation through time or space (four degrees, one per dimension); reflection through a plane (three degrees, the freedom in orientation of this plane); or a **boost** in any of the three spatial directions (three degrees). Composition of transformations is the operator of the **Poincaré group**, with proper rotations being produced as the composition of an even number of reflections. In classical physics, the **Galilean group** is a comparable ten-parameter group that acts on absolute time and space. Instead of **boosts**, it features shear mappings to relate co-moving frames of reference. [7]

Topology is concerned with the properties of space that are preserved under continuous deformations.

Compactness is a property that generalizes the notion of a subset of Euclidean space being **closed** (that is, containing all its limit points) and **bounded** (that is, having all its points lie within some fixed distance of each other).

A **manifold** is a topological space that locally resembles Euclidean space near each point.

In differential geometry, the **holonomy** of a connection on a smooth manifold is a general geometrical consequence of the curvature of the connection measuring the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported.

Two loops are **homotopic** if they can be continuously deformed into one another.

Homotopy allows us to define classes of topologically equivalent loops. [10, 11]

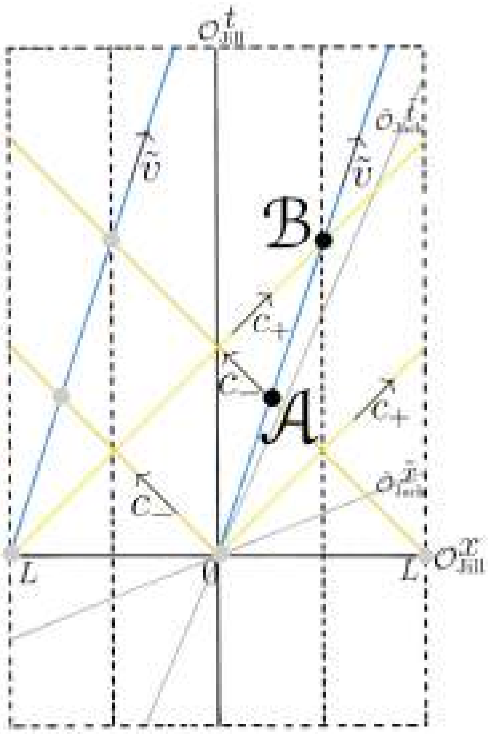
B Matlab Code for the System in Section 3.6

```
%%      Variables
%       L = x
%       vtilde = y
%%      Measurables
%       tau(A-) = A
%       tau(B+) = B
%%      Solver
                syms x y A B
%   Assumptions
    assume(x,'clear')
    assume(y,'clear')
    assume(A,'clear')
    assume(B,'clear')
    assume(x>0 & A>0 & B>0 & -1<y<1)
%   System To Solve
    eqn1 = x*((1-y^(2))^(1/2))/(1+y) - A == 0;
    eqn2 = x*((1-y^(2))^(1/2))/(1-y) - B == 0;
%   Solutions
    [solx,soly] = solve(eqn1,eqn2);
X = simplify(solx, 'IgnoreAnalyticConstraints', true, 'Steps', 100)
Y = simplify(soly, 'IgnoreAnalyticConstraints', true, 'Steps', 100)
%               Output
X =
(A*B)^(1/2)
Y =
-(A - B)/(A + B)
```

C Matlab Code for the System in Section 3.7

```
%%      Variables
%       v' = x
%       v'' = y
%       L = z
%%      Measurables
%       v = A
%       tau(Jill) = B
%       tau(Jack) = C
%%      Solver
                syms x y z A B C
%   Assumptions
        assume(x,'clear')
        assume(y,'clear')
        assume(z,'clear')
        assume(A,'clear')
        assume(B,'clear')
        assume(C,'clear')
    assume(z>0 & B>0 & C>0 & -1<x<1 & -1<y<1 & -1<A<1)
    eqn1 = (x+y)/(1+x*y) - A == 0;
    eqn2 = (z*(1-x^(2))^(1/2))/(x+y) - B == 0;
    eqn3 = (z*(1-y^(2))^(1/2))/(x+y) - C == 0;
%               Solutions
[solx,soly,solz] = solve(eqn1,eqn2,eqn3);
  X =  simplify(solx,'IgnoreAnalyticConstraints', true, 'Steps', 100)
  Y =  simplify(soly,'IgnoreAnalyticConstraints', true, 'Steps', 100)
  Z =  simplify(solz,'IgnoreAnalyticConstraints', true, 'Steps', 100)
%               Output
X =
 (C + B*(1 - A^2)^(1/2))/(A*C)
Y =
 (B + C*(1 - A^2)^(1/2))/(A*B)
Z =
 (A^2*B^2 - B^2 + A^2*C^2 - C^2 - 2*B*C*(1 - A^2)^(1/2))^(1/2)/(1 - A^2)^(1/2)
```

D Calculations for the System in Section 3.6



$x = \tilde{v}t$

Event A: $x = -t + L$

Event B: $x = t - L$

Coordinates for A: $\tilde{v}t = -t + L$

$t(\tilde{v}+1) = L$

$t = \frac{L}{\tilde{v}+1}$ $\therefore x = -\frac{L}{\tilde{v}+1} + L = \frac{L}{1} - \frac{L}{\tilde{v}+1} = \frac{(\tilde{v}+1)L - L}{\tilde{v}+1} = \frac{\tilde{v}L}{\tilde{v}+1}$

$\therefore A \ \frac{1}{\tilde{v}+1}\begin{pmatrix} L \\ \tilde{v}L \end{pmatrix}$

Coordinates for B: $\tilde{v}t = t - L$

$t(\tilde{v}-1) = -L$

$t = -\frac{L}{\tilde{v}-1} = \frac{L}{1-\tilde{v}}$, $x = \frac{L}{1-\tilde{v}} - L = \frac{L-(1-\tilde{v})L}{(1-\tilde{v})} = \frac{\tilde{v}L}{(1-\tilde{v})}$

$\therefore B \ \frac{1}{1-\tilde{v}}\begin{pmatrix} L \\ \tilde{v}L \end{pmatrix}$

Proper times:

$$\Delta\tau_A^2 = \Delta t^2 - \Delta x^2 = \left(\frac{L}{\tilde{v}+1}\right)^2 - \left(\frac{\tilde{v}L}{\tilde{v}+1}\right)^2 = \frac{L^2}{(\tilde{v}+1)^2} - \frac{\tilde{v}^2L^2}{(\tilde{v}+1)^2} = \frac{L^2(1-\tilde{v}^2)}{(\tilde{v}+1)^2}$$

$$\therefore \tau_{A,-} = \frac{L}{\tilde{v}+1}\sqrt{1-\tilde{v}^2}$$

$$\Delta\tau_B^2 = \Delta t^2 - \Delta x^2 = \left(\frac{L}{1-\tilde{v}}\right)^2 - \left(\frac{\tilde{v}L}{1-\tilde{v}}\right)^2 = \frac{L^2}{(1-\tilde{v})^2} - \frac{\tilde{v}^2L^2}{(1-\tilde{v})^2} = \frac{L^2(1-\tilde{v}^2)}{(1-\tilde{v})^2}$$

$$\therefore \tau_{B,+} = \frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}}$$

Define Ratio $\hat{\tau} = \frac{\tau_A}{\tau_B} = \frac{L\sqrt{1-\tilde{v}^2}}{\tilde{v}+1} \Big/ \frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}} = \frac{1-\tilde{v}}{1+\tilde{v}}$

$(\tilde{v}+1)\hat{\tau} = 1-\tilde{v}$

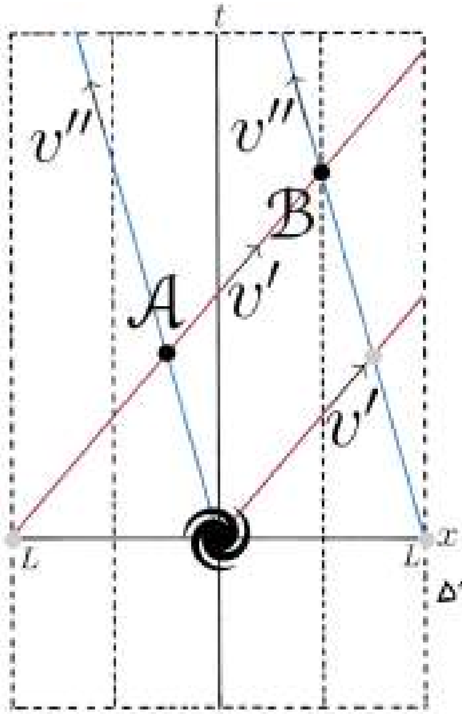
$\tilde{v}\hat{\tau} + \hat{\tau} = 1-\tilde{v}$

$\tilde{v}(\hat{\tau}+1) = 1-\hat{\tau}$ $\quad \therefore \tilde{v} = \frac{1-\hat{\tau}}{\hat{\tau}+1}$

$$\tau_A\tau_B = \left(\frac{L\sqrt{1-\tilde{v}^2}}{1+\tilde{v}}\right)\left(\frac{L\sqrt{1-\tilde{v}^2}}{1-\tilde{v}}\right) = \frac{L^2(1-\tilde{v}^2)}{(1-\tilde{v}^2)} = L^2$$

$$\therefore L = \sqrt{\tau_{A,-}\tau_{B,+}}$$

E Calculations for the System in Section 3.7



Find A coordinates

Jack $x = -v''t$

Jill $x = v't - L$

$$-v''t = v't - L$$

$$L = v't + v''t$$

$$L = t(v' + v'')$$

$$t = \frac{L}{v' + v''} \quad , x = -v''\frac{L}{v'+v''} = -\frac{v''L}{v'+v''}$$

$$A\begin{pmatrix} L/v'+v'' \\ -v''L/v'+v'' \end{pmatrix}$$

Proper time Jack measures between event A and $\binom{0}{0}$

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2$$

$$\Delta\tau^2_{A,\text{JACK}} = \left(\frac{L}{v'+v''}\right)^2 - \left(-\frac{v''L}{v'+v''}\right)^2 = \frac{L^2}{(v'+v'')^2} - \frac{v''^2L^2}{(v'+v'')^2} = \frac{L^2(1-v''^2)}{(v'+v'')^2}$$

① : $\tau_{A,\text{JACK}} = \frac{L\sqrt{1-v''^2}}{v'+v''}$ (Proper time between two successive meeting with Jill)

Velocity formula : $v = \frac{u+v}{1+uv}$, $u = \frac{v-w}{1-vw}$

③ $v = \frac{v'+v''}{1+v'v''}$

Proper time Jill measures between event A and $\binom{0}{-L}$

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2$$

$$\Delta\tau^2_{A,\text{JILL}} = \left(\frac{L}{v'+v''}\right)^2 - \left(\frac{v''L}{v'+v''} + L\right)^2 = \frac{L^2(1-v'^2)}{(v'+v'')^2}$$ (Proper time between two successive meeting with Jack)

② : $\tau_{A,\text{JILL}} = \frac{L(1-v'^2)}{(v'+v'')}$

$$v = \frac{v' + v''}{1 + v'v''} \quad (1), \quad t_{AJILL} = \frac{L\sqrt{1-v'^2}}{v'+v''} \quad (2), \quad t_{ATTACK} = \frac{L\sqrt{1-v''^2}}{v'+v''} \quad (3)$$

$$A = \frac{x+y}{1+xy}, \quad B = \frac{z\sqrt{1-x}}{x+y}, \quad C = \frac{z\sqrt{1-y}}{x+y}$$

Try

$$\frac{B^2}{C^2} = \frac{z^2(1-x)}{(x+y)^2}\frac{(x+y)^2}{z^2(1-y)} = \frac{1-x}{1-y}$$

$$\left(\frac{B}{C}\right)^2(1-y) = 1-x$$

$$(4) \quad x = 1 + \left(\frac{B}{C}\right)^2(y-1)$$

Try (4) into (1)

$$A(1+xy) = x+y$$

$$A + Axy - x = y$$

$$x(Ay-1) = y-A$$

$$(5) \quad x = \frac{y-A}{Ay-1}$$

(4) = (5)

$$1 + \left(\frac{B}{C}\right)^2(y-1) = \frac{y-A}{Ay-1}$$

$$Ay - 1 + \left(\frac{B}{C}\right)^2(y-1)(Ay-1) = y-A$$

$$Ay - 1 + \left(\frac{B}{C}\right)^2(Ay^2 - y - Ay + 1) = y-A$$

$$y^2\underbrace{\left\{A\left(\frac{B}{C}\right)^2\right\}}_{X} + y\underbrace{\left\{A - A\left(\frac{B}{C}\right)^2 - \left(\frac{B}{C}\right)^2 - 1\right\}}_{Y} + \underbrace{\left\{A - 1 + \left(\frac{B}{C}\right)^2\right\}}_{Z} = 0$$

$$x = 1 + \left(\frac{B}{C}\right)^2\left(\frac{-Y \pm \sqrt{Y^2 - 4XZ}}{2X} - 1\right) \qquad y = \frac{-Y \pm \sqrt{Y^2-4XZ}}{2X}$$

$$z = \frac{C\left\{1 + \left(\frac{B}{C}\right)^2\left(\frac{-Y \pm \sqrt{Y^2-4XZ}}{2X} - 1\right) + \frac{-Y \pm \sqrt{Y^2-4XZ}}{2X}\right\}}{\sqrt{1 - \frac{-Y \pm \sqrt{Y^2-4XZ}}{2X}}}$$

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